On the solutions of linear differential equations with polynomial coefficients

Khalid Alshammari

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Signature(Second Reader) Date

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Signature(Dean of Science) Date

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University of Prince Edward Island
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To my family, who have gotten me this far.

...

Abstract

The problem addressed in this thesis reads as follows:

Given a general differential equation

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + \rho(x)y = 0,$$

where $p(x)$, $q(x)$ and $\rho(x)$ are polynomials of the variable x and defined on a finite or infinite interval, what conditions on the polynomial coefficients for which at least one of the series solutions of the differential equation truncates to a polynomial?

The present thesis attempts to answer this question for various classes of second-order differential equations.

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Contents

Preface

Consider

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + \rho(x)y = 0,$$

where $p(x)$, $q(x)$ and $\rho(x)$ are polynomials defined on a finite or infinite interval. The classical theory of ordinary differential equations guarantee, for the polynomials $p(x) > 0, q(x)$, $\rho(x)$ and in the neighbourhood of the ordinary point $x = 0$, the existence of two linearly independent solutions (neither is a constant multiple of the other):

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x),$$

where

$$y_1(x) = \sum_{k=0}^{\infty} \alpha_k x^k, \qquad y_2(x) = \sum_{k=0}^{\infty} \beta_k x^k,$$

and c_1 and c_2 are constants. On other hand, if $x = x_0$ is a singular point ($p(x_0) = 0$) and the products $(x - x_0)q(x)/p(x)$ and $(x - x_0)^2\rho(x)/p(x)$ are both analytic at x_0, then there at least one power series (Frobenius) solution of the differential equation.

In the present thesis, we are interested in the case where one of the infinite series solutions $y_1(x)$ and $y_2(x)$ truncates to a finite number of terms,

$$y_n(x) = \sum_{k=0}^{n} \alpha_k x^k,$$

for some non-negative integers $n \geq 0$.

To this end, we examine various classes of the second-order differential equations with polynomial coefficients by deriving necessary and sufficient conditions to guarantee the existence of such polynomial solutions.

The thesis divided into nine chapters: The first chapter review the general theory of series solutions of the second order differential equation in the neighborhood of an ordinary point and the neighborhood of a singular point. The second part of this chapter deals with some transformations of general differential equations that allows expressing the solutions of the transformed differential equations in term of the original solutions. We give four theorems; the most general transformations listed in Theorems (1.2.2) and (1.2.3), both theorems are proved. The underlying idea of Theorem (1.2.1) is to make a change of variable to the independent variable. The Theorems (1.2.2) and (1.2.3) deal with the transformations of both the independent and dependent variables. Theorem (1.2.4) deals with a differential transformation of the differential equation.

Chapters 2-9 deal with a particular class of differential equations that frequently appears in the mathematical physics literature. Many of those differential equations generalized the classical equations before-mentioned as Laguerre, Legendre, Jacobi, etc. In particular, Chapters 2 to 6 deal with the equation

$$(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + (\beta_0 + \beta_1 x)y' + \varepsilon_0 y = 0$$

wherein we give the most general series solutions $y(x) = \sum_{k=0}^{\infty} C_k x^k$ using two recurrence relation to evaluating the coefficients f$\{C_k\}$ or all k.

The conditions on the real parameters $\alpha_0, \alpha_1, \alpha_2$, β_0, β_1 and ε_0 that ensure the existence of polynomial solutions are discussed in detail. As in applications, we discuss some of the classical equations that follow naturally from our generalization.

In Chapters 7 to 9, we generalize our approach to study

$$P_j(x)y'' + P_{j-1}(x)y' + P_{j-2}(x)y = 0$$

where P_j are polynomials of degree $j = 3, 4$. In each case, we give the general series solutions in the neighborhood of an ordinary points or a singular points. We also examine the conditions on the polynomial coefficients to establish the polynomial solutions.

Chapter 1

Introduction

Ordinary differential equations have been a major research area of pure and applied mathematics since the early work of Isaac Newton and Gottfried Leibniz in the mid 17th century [1]. In the 19th century, the general theory was enriched by the major development of understanding the existence theorems. Although many aspects of differential equations were well-studied [2, 3], the general theory remains an important field of on-going investigation especially with the new development of many applications in mathematics-physics. More exciting new types of differential equations and the demand for general and particular solutions have appeared, for example, in special functions and orthogonal polynomials.

It has been noted that the study of many physical phenomena reduces to the analysis of certain differential equations: these, in turn, demand insight for understanding their general solutions. One of the simplest and most widely used linear differential equation [4] is given by

$$P_2(x)y''(x) + P_1(x)y'(x) + P_0(x)y(x) = 0,$$

where P_j is a polynomial of degree j. This equation and many of its various forms frequently appear in all different sub-fields of physics. The exact solutions of many particular cases of this equation and many other related equations can be found in two classical works [5, 6]. A general form of this equation and further generalization shall be our focus in this work. In the next section, we review the general theory to obtain series solutions of the second-order differential equations.

1.1 Power series solutions

We briefly review the power series solutions of second-order linear differential equations (for further details see [7, 8, 9, 10]). Consider the second-order linear differential equation,

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + \rho(x)y = 0, \tag{1.1.1}$$

around $x = x_0$, where $p(x)$, $q(x)$ and $\rho(x)$ are polynomials of the independent variables x defined on $-\infty \leq a < b \leq \infty$.

There are three different cases, depending of the behaviour of $p(x), q(x)$ and $\rho(x)$ and, in which $x = x_0$ is classified as an *ordinary point*, a *regular singular point*, or an *irregular singular point.*

Definition 1.1.1. If $p(x_0) \neq 0$, and if the ratios $q(x)/p(x)$ and $\rho(x)/p(x)$ are analytic at $x = x_0$, then $x = x_0$ is called an **ordinary point.**

By definition the function $f(x)$ is analytic if $f(x)$ has a convergent power series with nonzero radius of convergence. For example,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad -\infty < x < \infty$$

is an analytic function with radius of convergence $R = \infty$.

Definition 1.1.2. If x_0 is not an ordinary point of (1.1.1), it is a *singular point.*

Theorem 1.1.1. *(L. I. Fuchs [12]): If x_0 is an ordinary point of the equation* (1.1.1)*, then the general solution of the differential equation* (1.1.1) *can be expressed as*

$$y(x) = C_0 + C_1 \cdot (x - x_0) + C_2 \cdot (x - x_0)^2 + \cdots = \sum_{k=0}^{\infty} C_k \cdot (x - x_0)^k, \tag{1.1.2}$$

where C_0 and C_1 are arbitrary constants. The radius of convergence of any power series solution of the form given above is at least as large as the distance from x_0 to the nearest singular point of the given differential equation.

Definition 1.1.3. Let x_0 be a singular point, if the products

$$(x - x_0)q(x)/p(x) \quad \text{and} \quad (x - x_0)^2\rho(x)/p(x)$$

are both analytic at the singular point x_0, then the point x_0 is called a **regular singular point** of equation (1.1.1).

Definition 1.1.4. A point x_0 is called an **irregular singular point** of (1.1.1) if it is neither an ordinary point nor a regular singular point.

A simple criterion for the regularity of the ratios $(x - x_0)q(x)/p(x)$ and $(x - x_0)^2\rho(x)/p(x)$ at the singular point x_0 is that:
Both of the limits

$$\lim_{x \to x_0} \frac{(x - x_0)q(x)}{p(x)} = P_0, \qquad \text{and} \qquad \lim_{x \to x_0} \frac{(x - x_0)^2\rho(x)}{p(x)} = Q_0, \tag{1.1.3}$$

exist and finite.

Theorem 1.1.2. *(L. I. Fuchs [12]) If the point x_0 is a regular singular point of the differential equation* (1.1.1)*, and that the indicial equation $r(r-1) + P_0 r + Q_0 = 0$ has two distinct roots r_1 and r_2 where $r_1 - r_2$ is not an integer, then the equation* (1.1.1) *has two different formal power series expansions solutions in the form*

$$y_1(x) = (x - x_0)^{r_1} \cdot \sum_{k=0}^{\infty} C_k^{r_1}(x - x_0)^k, \quad (0 < |x - x_0| < R; C_0^{r_1} \neq 0) \tag{1.1.4}$$

and

$$y_2(x) = (x - x_0)^{r_2} \cdot \sum_{k=0}^{\infty} C_k^{r_2}(x - x_0)^k, \quad (0 < |x - x_0| < R; C_0^{r_2} \neq 0) \tag{1.1.5}$$

that constitute the fundamental system of solutions.

Theorem 1.1.3. *(L. I. Fuchs [12]) Let x_0 be a regular singular point of the differential equation* (1.1.1)*. Suppose the exponents, for x_0, of the indicial equation, r_1 and r_2, are such that $r_1 - r_2 = n \in \{0, 1, 2, \dots\}$. Then the differential equation possesses a fundamental system of solutions in the form*

$$y_1(x) = (x - x_0)^{r_2} \sum_{k=0}^{\infty} C_k^{r_2} \cdot (x - x_0)^k, \quad (0 < |x - x_0| < R; C_0^{r_2} \neq 0)$$

and if $n = 0$, the second linearly independent solution is

$$y_2(x) = g_0\, y_1(x) \ln(x - x_0) + (x - x_0)^{r_2} \sum_{k=1}^{\infty} C_k^{r_2} \cdot (x - x_0)^k,$$

and if $n = 1, 2, \dots$ (positive integer),the second linearly independent solution is

$$y_2(x) = g_n\, y_1(z) \ln(x - x_0) + (x - x_0)^{r_2} \sum_{k=0}^{\infty} C_k^{r_2} \cdot (x - x_0)^k, \qquad C_0^{r_2} \neq 0,$$

where g_n may or may not equal zero.

1.2 Elementary transformations

In this section, we discuss several transformations that relates different classes of linear differential equations. These transformations play fundamental roles in mathematical physics applications [13, 14].

Theorem 1.2.1. *Given a general second-order homogeneous differential equation*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0, \tag{1.2.1}$$

the change the independent variable from x to $z = z(x)$ transforms the equation (1.2.1) *into*

$$\frac{d^2y(z)}{dz^2} + \left(\frac{z''(x) + p(x)}{z'(x)}\right)\frac{dy(z)}{dz} + \frac{q(x)}{(z'(x))^2}y(z) = 0. \tag{1.2.2}$$

Furthermore, equation (1.2.2) *transformed into an equation with constant coefficients if the ratio $(q'(x) + 2p(x)q(x))/(q(x))^{3/2}$ is constant.*

Proof. Using the Chain rule, the change of independent variable $z = z(x)$ gives,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dz}\frac{dz}{dx} = z'(x)\frac{dy}{dz},\\
\frac{d^2y}{dx^2} &= \frac{d}{dz}\left(z'(x)\frac{dy}{dz}\right)\frac{dz}{dx} = z'(x)\left(z''(x)\frac{dy}{dz} + z'(x)\frac{d^2y}{dz^2}\right)\\
&= z'(x)z''(x)\frac{dy}{dz} + (z'(x))^2\frac{d^2y}{dz^2}.
\end{aligned}$$

Direct substitutions of these identities back into the differential equation (1.2.1) gives the required results. The equation (1.2.2) becomes a differential equation with constant coefficients if

$$\frac{q(x)}{(z'(x))^2} = A_1, \qquad \frac{z''(x) + p(x)}{z'(x)} = A_2$$

The first equation gives

$$\frac{q(x)}{A_1} = (z'(x))^2 \Longrightarrow \frac{q'(x)}{A_1} = 2(z'(x))^2 z''. \Longrightarrow \frac{q'(x)}{2(z'(x))^2 A_1} = z''.$$

Substituting into $(z''(x) + p(x))/z'(x) = A_2$, or equivalently

$$\frac{\dfrac{q'(x)}{2(z'(x))^2 A_1} + p(x)}{z'(x)} = A_2 \Longrightarrow A_2 = \frac{q'(x) + 2(z'(x))^2 A_1 p(x)}{2(z'(x))^3 A_1} \Longrightarrow A_2 = \frac{q'(x) + 2q(x)p(x)}{2(q(x))^{3/2}/\sqrt{A_1}}$$

gives

$$\frac{q'(x) + 2p(x)q(x)}{(q(x))^{3/2}} = 2A_2/\sqrt{A_1} \qquad (Constant).$$

□

Theorem 1.2.2. ***(E.E. Kummer [11])*** *The change of variables*

$$y(x) = w(x)v(z(x)), \tag{1.2.3}$$

transforms the linear homogeneous differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0 \tag{1.2.4}$$

into

$$\frac{d^2v}{dz^2} + P(z)\frac{dv}{dx} + Q(z)v(z) = 0 \tag{1.2.5}$$

where

$$P(z) = \frac{2}{w(x)}\frac{\dfrac{dw}{dx}}{\left(\dfrac{dz}{dx}\right)} + \frac{\dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)^2} + \frac{p(x)}{\left(\dfrac{dz}{dx}\right)} \tag{1.2.6}$$

and

$$Q(z) = \frac{1}{w(x)}\frac{\dfrac{d^2w}{dx^2}}{\left(\dfrac{dz}{dx}\right)^2} + \frac{p(x)}{w(x)}\frac{\dfrac{dw}{dx}}{\left(\dfrac{dz}{dx}\right)^2} + \frac{q(x)}{\left(\dfrac{dz}{dx}\right)^2} \tag{1.2.7}$$

Here the function $w(x)$ satisfies the equation

$$[w(x)]^2 = C\frac{dx}{dz}\exp\left(\int P(z)dz - \int p(x)dx\right) \tag{1.2.8}$$

and the transformation $z = z(x)$ of the independent variable is given by solving the higher-order differential equation

$$\begin{aligned} &2\frac{d}{dz}\left(\frac{d^2z}{dx^2}\right) - 3\left(\frac{d}{dz}\left(\frac{dz}{dx}\right)\right)^2 \\ &\qquad - \left(2\frac{dP}{dz} + P^2(z) - 4Q(z)\right)\left(\frac{dz}{dx}\right)^2 + 2\frac{dp}{dx} + p^2(x) - 4q(x) = 0 \end{aligned} \tag{1.2.9}$$

Proof. For $y(x) = w(x)v(z(x))$, we note

$$\frac{dy}{dx} = \frac{dw}{dx}v(z(x)) + w(x)\frac{dv}{dz}\frac{dz}{dx},$$

and

$$\frac{d^2y}{dx^2} = \frac{d^2w}{dx^2}v(z(x)) + 2\frac{dw}{dx}\frac{dv}{dz}\frac{dz}{dx} + w(x)\frac{d^2v}{dz^2}\left(\frac{dz}{dx}\right)^2 + w(x)\frac{dv}{dz}\frac{d^2z}{dx^2}.$$

Substituting in the given differential equation we obtain

$$\begin{aligned}\frac{d^2w}{dx^2}v(z(x)) + 2\frac{dw}{dx}\frac{dv}{dz}\frac{dz}{dx} + w(x)\frac{d^2v}{dz^2}\left(\frac{dz}{dx}\right)^2 + w(x)\frac{dv}{dz}\frac{d^2z}{dx^2}&\\ + p(x)\left(\frac{dw}{dx}v(z(x)) + w(x)\frac{dv}{dz}\frac{dz}{dx}\right) + q(x)w(x)v(z(x)) = 0.&\end{aligned}$$

This equation is easily reduced to

$$\begin{aligned}\left[w(x)\left(\frac{dz}{dx}\right)^2\right]\frac{d^2v}{dz^2} + \left[2\frac{dw}{dx}\frac{dz}{dx} + w(x)\frac{d^2z}{dx^2} + p(x)w(x)\frac{dz}{dx}\right]\frac{dv}{dz}&\\ + \left(\frac{d^2w}{dx^2} + p(x)\frac{dw}{dx} + q(x)w(x)\right)v(z(x)) = 0&\end{aligned}$$

By comparison

$$P(z) = \frac{\left[2\frac{dw}{dx}\frac{dz}{dx} + w(x)\frac{d^2z}{dx^2} + p(x)w(x)\frac{dz}{dx}\right]}{\left[w(x)\left(\frac{dz}{dx}\right)^2\right]} = \frac{2}{w(x)}\frac{\frac{dw}{dx}}{\left(\frac{dz}{dx}\right)} + \frac{\frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} + \frac{p(x)}{\left(\frac{dz}{dx}\right)}.$$

This equation can be written as

$$P(z)\left(\frac{dz}{dx}\right) = 2\frac{d}{dx}\log(w(x)) + \frac{d}{dx}\log\left(\frac{dz}{dx}\right) + p(x).$$

Integrating both sides with respect to the variable x yields

$$\int P(z)dz = 2\log(w(x)) + \log\left(\frac{dz}{dx}\right) + \int p(x)dx + \log C$$

for some constant C. Solving this equation for $w(x)$, we obtain

$$[w(x)]^2 = C\,\frac{dx}{dz}\,\exp\left(\int P(z)dz - \int p(x)dx\right).$$

Differentiating with respect to x,

$$2w(x)\frac{dw}{dx} = C\,\frac{d}{dx}\left(\frac{1}{\frac{dz}{dx}}\right)\exp\left(\int P(z)dz - \int p(x)dx\right)$$

$$
\begin{aligned}
&+ C\frac{dx}{dz}\frac{d}{dx}\left[\exp\left(\int P(z)dz - \int p(x)dx\right)\right]\\
&= C\left(-\frac{\dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)^2}\right)\exp\left(\int P(z)dz - \int p(x)dx\right)\\
&+ C\frac{dx}{dz}\exp\left(\int P(z)dz - \int p(x)dx\right)\frac{d}{dx}\left[\left(\int P(z)dz - \int p(x)dx\right)\right]\\
&= -\left(\frac{\dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)}\right)w^2(x) + w^2(x)\frac{d}{dx}\left[\left(\int P(z)dz - \int p(x)dx\right)\right]
\end{aligned}
$$

Therefore

$$
\frac{dw}{dx} = -\frac{1}{2}\left(\frac{\dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)}\right)w(x) + \frac{w(x)}{2}\frac{d}{dz}\left[\int P(z)dz\right]\frac{dz}{dx} - \frac{1}{2}p(x).
$$

or

$$
\frac{1}{w(x)}\frac{dw}{dx} = -\frac{1}{2}\frac{d}{dx}\log\left(\frac{dz}{dx}\right) + \frac{1}{2}P(z)\frac{dz}{dx} - \frac{1}{2}p(x).
$$

A further differentiation of this equation with respect to x implies

$$
\frac{1}{w(x)}\frac{d^2w}{dx^2} - \left(\frac{1}{w(x)}\frac{dw}{dx}\right)^2 = -\frac{1}{2}\frac{d^2}{dx^2}\log\left(\frac{dz}{dx}\right) + \frac{1}{2}\frac{dP}{dz}\left(\frac{dz}{dx}\right)^2 - \frac{1}{2}\frac{dp}{dx}
$$

or

$$
\begin{aligned}
\frac{1}{w(x)}\frac{d^2w}{dx^2} &= \left(-\frac{1}{2}\frac{d}{dx}\log\left(\frac{dz}{dx}\right) + \frac{1}{2}P(z)\frac{dz}{dx} - \frac{1}{2}p(x)\right)^2\\
&\quad - \frac{1}{2}\frac{d^2}{dx^2}\log\left(\frac{dz}{dx}\right) + \frac{1}{2}\frac{dP}{dz}\left(\frac{dz}{dx}\right)^2 - \frac{1}{2}\frac{dp}{dx}.
\end{aligned}
$$

Using the expression for $Q(z)$,

$$
Q(z)\left(\frac{dz}{dx}\right)^2 = \frac{1}{w(x)}\frac{d^2w}{dx^2} + \frac{p(x)}{w(x)}\frac{dw}{dx} + q(x),
$$

it yields

$$
\begin{aligned}
Q(z)\left(\frac{dz}{dx}\right)^2 &= \left(-\frac{1}{2}\frac{d}{dx}\log\left(\frac{dz}{dx}\right) + \frac{1}{2}P(z)\frac{dz}{dx} - \frac{1}{2}p(x)\right)^2\\
&\quad - \frac{1}{2}\frac{d^2}{dx^2}\log\left(\frac{dz}{dx}\right) + \frac{1}{2}\frac{dP}{dz}\left(\frac{dz}{dx}\right)^2 - \frac{1}{2}\frac{dp}{dx}\\
&\quad + p(x)\left(-\frac{1}{2}\frac{d}{dx}\log\left(\frac{dz}{dx}\right) + \frac{1}{2}P(z)\frac{dz}{dx} - \frac{1}{2}p(x)\right) + q(x)
\end{aligned}
$$

which is now free of $w(x)$. Expand the right hand side and simplify to complete the proof of the theorem.

□

Theorem 1.2.3. ***(E.E. Kummer [11])*** *The change of variables*

$$y(x) = w(x)v(z(x)) \tag{1.2.10}$$

transforms the linear homogeneous differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0 \tag{1.2.11}$$

into

$$\frac{d^2w}{dx^2} + \mathcal{P}(x)\frac{dw}{dx} + \mathcal{Q}(x)w(x) = 0 \tag{1.2.12}$$

where

$$\mathcal{P}(x) = \frac{2}{v(z(x))}\frac{dv}{dz}\frac{dz}{dx} + p(x) \tag{1.2.13}$$

and

$$\mathcal{Q}(x) = q(x) + \frac{1}{2}\left(\frac{d\mathcal{P}}{dx} - \frac{dp}{dx}\right) + \frac{1}{4}\left(\mathcal{P}(x) - p(x)\right)\left(\mathcal{P}(x) + p(x)\right) \tag{1.2.14}$$

Proof. Starting with $y(x) = w(x)v(z(x))$, we note

$$\frac{dy}{dx} = \frac{dw}{dx}v(z(x)) + w(x)\frac{dv}{dz}\frac{dz}{dx},$$

and

$$\frac{d^2y}{dx^2} = \frac{d^2w}{dx^2}v(z(x)) + 2\frac{dw}{dx}\frac{dv}{dz}\frac{dz}{dx} + w(x)\frac{d^2v}{dz^2}\left(\frac{dz}{dx}\right)^2 + w(x)\frac{dv}{dz}\frac{d^2z}{dx^2}.$$

Substituting in the given differential equation we obtain

$$\begin{aligned} v(z(x))\frac{d^2w}{dx^2} &+ \left[2\frac{dv}{dz}\frac{dz}{dx} + p(x)v(z(x))\right]\frac{dw}{dx} \\ &+ \left[\frac{d^2v}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dv}{dz}\frac{d^2z}{dx^2} + p(x)\frac{dv}{dz}\frac{dz}{dx} + q(x)v(z(x))\right]w(x) = 0. \end{aligned}$$

That can be written as

$$\frac{d^2w}{dx^2} + \mathcal{P}(x)\frac{dw}{dx} + \mathcal{Q}(x)w(x) = 0$$

where

$$\begin{aligned} \mathcal{P}(x) &= \frac{2\dfrac{dv}{dz}\dfrac{dz}{dx} + p(x)v(z(x))}{v(z(x))} = \frac{2}{v(z(x))}\frac{dv}{dz}\frac{dz}{dx} + p(x) \\ \mathcal{Q}(x) &= \frac{\left(\dfrac{dz}{dx}\right)^2\dfrac{d^2v}{dz^2} + \left(\dfrac{d^2z}{dx^2} + p(x)\dfrac{dz}{dx}\right)\dfrac{dv}{dz} + q(x)v(z(x))}{v(z(x))} \\ &= \left(\frac{dz}{dx}\right)^2\frac{1}{v(z(x))}\frac{d^2v}{dz^2} + \left(\frac{d^2z}{dx^2} + p(x)\frac{dz}{dx}\right)\frac{1}{v(z(x))}\frac{dv}{dz} + q(x) \\ &= \left(\frac{dz}{dx}\right)^2\frac{d}{dz}\left(\frac{1}{v(z(x))}\frac{dv}{dz}\right) + \left(\frac{dz}{dx}\right)^2\left(\frac{1}{v(z(x))}\frac{dv}{dz}\right)^2 + \left(\frac{d^2z}{dx^2} + p(x)\frac{dz}{dx}\right)\frac{1}{v(z(x))}\frac{dv}{dz} + q(x). \end{aligned}$$

However, since

$$\frac{1}{2}\frac{(\mathcal{P}(x)-p(x))}{\dfrac{dz}{dx}}=\frac{1}{v(z(x))}\frac{dv}{dz},$$

we finally obtain

$$\begin{aligned}\mathcal{Q}(x)-q(x)&=\left(\frac{dz}{dx}\right)^2\frac{d}{dz}\left(\frac{1}{2}\frac{(\mathcal{P}(x)-p(x))}{\dfrac{dz}{dx}}\right)\\&\quad+\left(\frac{dz}{dx}\right)^2\left(\frac{1}{2}\frac{(\mathcal{P}(x)-p(x))}{\dfrac{dz}{dx}}\right)^2+\left(\frac{d^2z}{dx^2}+p(x)\frac{dz}{dx}\right)\left(\frac{1}{2}\frac{(\mathcal{P}(x)-p(x))}{\dfrac{dz}{dx}}\right)\end{aligned}$$

that yields

$$\begin{aligned}\mathcal{Q}(x)-q(x)&=\left(\frac{dz}{dx}\right)^2\left(\frac{1}{2}\frac{\dfrac{dz}{dx}\left(\dfrac{d\mathcal{P}}{dx}-\dfrac{dp}{dx}\right)\dfrac{dx}{dz}-(\mathcal{P}(x)-p(x))\dfrac{d^2z}{dx^2}\dfrac{dx}{dz}}{\left(\dfrac{dz}{dx}\right)^2}\right)\\&\quad+\left(\frac{dz}{dx}\right)^2\left(\frac{1}{2}\frac{(\mathcal{P}(x)-p(x))}{\dfrac{dz}{dx}}\right)^2+\left(\frac{d^2z}{dx^2}+p(x)\frac{dz}{dx}\right)\left(\frac{1}{2}\frac{(\mathcal{P}(x)-p(x))}{\dfrac{dz}{dx}}\right).\end{aligned}$$

Further simplification also yields

$$\begin{aligned}\mathcal{Q}(x)-q(x)&=\frac{1}{2}\left(\frac{d\mathcal{P}}{dx}-\frac{dp}{dx}\right)-\frac{1}{2}(\mathcal{P}(x)-p(x))\frac{d^2z}{dx^2}\frac{dx}{dz}\\&\quad+\frac{1}{4}(\mathcal{P}(x)-p(x))^2+\frac{1}{2}\left(\frac{\dfrac{d^2z}{dx^2}}{\dfrac{dz}{dx}}+p(x)\right)(\mathcal{P}(x)-p(x))\end{aligned}$$

whence

$$\begin{aligned}\mathcal{Q}(x)-q(x)&=\frac{1}{2}\left(\frac{d\mathcal{P}}{dx}-\frac{dp}{dx}\right)-\frac{1}{2}(\mathcal{P}(x)-p(x))\frac{d}{dx}\log\frac{dx}{dz}\\&\quad+\frac{1}{4}(\mathcal{P}(x)-p(x))^2+\frac{1}{2}\left(\frac{d}{dx}\log\frac{dz}{dx}+p(x)\right)(\mathcal{P}(x)-p(x)).\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{Q}(x)-q(x)&=\frac{1}{2}\left(\frac{d\mathcal{P}}{dx}-\frac{dp}{dx}\right)-\frac{1}{2}\mathcal{P}(x)\frac{d}{dx}\log\frac{dx}{dz}+\frac{1}{2}p(x)\frac{d}{dx}\log\frac{dx}{dz}\\&\quad+\frac{1}{4}(\mathcal{P}(x)-p(x))^2+\frac{1}{2}\mathcal{P}(x)\frac{d}{dx}\log\frac{dz}{dx}-\frac{1}{2}p(x)\frac{d}{dx}\log\frac{dz}{dx}\\&\quad+\frac{1}{2}p(x)(\mathcal{P}(x)-p(x)),\\&=\frac{1}{2}\left(\frac{d\mathcal{P}}{dx}-\frac{dp}{dx}\right)+\frac{1}{4}(\mathcal{P}(x)-p(x))^2+\frac{1}{2}p(x)(\mathcal{P}(x)-p(x))\end{aligned}$$

$$= \frac{1}{2}\left(\frac{d\mathcal{P}}{dx} - \frac{dp}{dx}\right) + \frac{1}{4}\left(\mathcal{P}(x) - p(x)\right)\left(\left(\mathcal{P}(x) - p(x)\right) + 2p(x)\right)$$

$$\mathcal{Q}(x) = q(x) + \frac{1}{2}\left(\frac{d\mathcal{P}}{dx} - \frac{dp}{dx}\right) + \frac{1}{4}\left(\mathcal{P}(x) - p(x)\right)\left(\mathcal{P}(x) + p(x)\right)$$

□

Corollary. *The change of variables*

$$y(x) = w(x)e^{z(x)}, \tag{1.2.15}$$

transforms the linear homogeneous differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0, \tag{1.2.16}$$

into

$$\frac{d^2w}{dx^2} + \left(2\frac{dz}{dx} + p(x)\right)\frac{dw}{dx} + \left(q(z) + \frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 + p(x)\frac{dz}{dx}\right)w(x) = 0. \tag{1.2.17}$$

Theorem 1.2.4. ***(Stanêk and Vosmansky [14])*** *The transformation*

$$z(x) = \beta(x)\,y'(x) + \alpha(x)\,y(x)$$

maps the solutions of the linear differential equation

$$y''(x) + A_1(x)\,y'(x) + B_1(x)\,y(x) = 0,$$

onto the solutions of the differential equation

$$z''(x) = \left(\frac{\mu_1(x)\alpha(x) - \mu_2(x)\beta(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)}\right)z'(x) + \left(\frac{\mu_2(x)\tau_1(x) - \mu_1(x)\tau_2(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)}\right)z(x).$$

where

$$\tau_1(x) = \alpha(x) + \beta'(x) - \beta(x)\,A_1(x), \qquad \tau_2(x) = \alpha'(x) - \beta(x)B_1(x),$$
$$\mu_1(x) = \tau_1'(x) + \tau_2(x) - \tau_1(x)A_1(x), \qquad \mu_2(x) = \tau_2'(x) - \tau_1(x)B_1(x),$$

if

$$\alpha^2(x) + B_1(x)\beta^2(x) - \beta(x)\alpha'(x) + \alpha(x)\beta(x) - A_1(x)\alpha(x)\beta(x) \neq 0$$

Proof. The derivative of the function $z(x) = \alpha(x)\,y(x) + \beta(x)\,y'(x)$ can be expressed in the form

$$\begin{aligned} z'(x) &= \alpha'(x)\,y(x) + \alpha(x)\,y'(x) + \beta'(x)\,y'(x) + \beta(x)\,y''(x) \\ &= \alpha'(x)\,y(x) + \alpha(x)\,y'(x) + \beta'(x)\,y'(x) + \beta(x)\,\left(-A_1(x)y'(x) - B_1(x)y(x)\right) \\ &= \left(\alpha(x) + \beta'(x) - \beta(x)\,A_1(x)\right)y'(x) + \left(\alpha'(x) - \beta(x)B_1(x)\right)y(x). \end{aligned}$$

Write this equation as

$$z'(x) = \tau_1(x)y'(x) + \tau_2(x)y(x),$$

where

$$\tau_1(x) = \alpha(x) + \beta'(x) - \beta(x)\,A_1(x), \qquad \tau_2(x) = \alpha'(x) - \beta(x)B_1(x).$$

We note

$$\begin{aligned} z''(x) &= \tau_1'(x)y'(x) + \tau_1(x)y''(x) + \tau_2'(x)y(x) + \tau_2(x)y'(x) \\ &= \tau_1'(x)y'(x) + \tau_1(x)(-A_1(x)y'(x) - B_1(x)y(x)) + \tau_2'(x)y(x) + \tau_2(x)y'(x) \\ &= (\tau_1'(x) + \tau_2(x) - \tau_1(x)A_1(x))y'(x) + (\tau_2'(x) - \tau_1(x)B_1(x))y(x) \\ &= \mu_1(x)\, y'(x) + \mu_2(x)\, y(x). \end{aligned}$$

where

$$\begin{aligned} \mu_1(x) &= \tau_1'(x) + \tau_2(x) - \tau_1(x)A_1(x) \\ &= \alpha'(x) + \beta''(x) - \beta'(x)A_1(x) - \beta(x)A_1'(x), \\ \mu_2(x) &= \tau_2'(x) - \tau_1(x)B_1(x) \\ &= \alpha''(x) - \beta'(x)B_1(x) - \beta(x)B_1'(x) - (\alpha(x) + \beta'(x) - \beta(x)\, A_1(x))B_1(x) \end{aligned}$$

From

$$\begin{pmatrix} z'(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} \tau_1(x) & \tau_2(x) \\ \beta(x) & \alpha(x) \end{pmatrix} \begin{pmatrix} y'(x) \\ y(x) \end{pmatrix}.$$

If $\tau_1(x)\alpha(x) - \beta(x)\tau_2(x) \neq 0$, we have

$$\begin{pmatrix} y'(x) \\ y(x) \end{pmatrix} = \frac{\begin{pmatrix} \alpha(x) & -\tau_2(x) \\ -\beta(x) & \tau_1(x) \end{pmatrix}}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)} \begin{pmatrix} z'(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha(x)z'(x) - \tau_2(x)z(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)} \\ \dfrac{-\beta(x)z'(x) + \tau_1(x)z(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)} \end{pmatrix}.$$

Then

$$y'(x) = \frac{\alpha(x)z'(x) - \tau_2(x)z(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)}, \qquad y(x) = \frac{-\beta(x)z'(x) + \tau_1(x)z(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)}.$$

Finally,

$$\begin{aligned} z''(x) &= \mu_1(x)\left(\frac{\alpha(x)z'(x) - \tau_2(x)z(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)}\right) + \mu_2(x)\left(\frac{-\beta(x)z'(x) + \tau_1(x)z(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)}\right) \\ &= \left(\frac{\mu_1(x)\alpha(x) - \mu_2(x)\beta(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)}\right) z'(x) + \left(\frac{\mu_2(x)\tau_1(x) - \mu_1(x)\tau_2(x)}{\tau_1(x)\alpha(x) - \beta(x)\tau_2(x)}\right) z(x). \end{aligned}$$

□

Chapter 2

On the solutions of the differential equation: $(1+\alpha\, x^2)\, y'' + \beta\, x\, y' + \gamma\, y = 0$

2.1 Series solutions

Theorem 2.1.1. *The coefficients* $\{c_k\}_{k=0}^{\infty}$ *of the infinite series solution* $y(x) = \sum_{k=0}^{\infty} c_k\, x^k$ *of the differential equation*

$$(1+\alpha\, x^2)\, y'' + \beta\, x\, y' + \gamma\, y = 0 \tag{2.1.1}$$

satisfy the two-term recurrence relation

$$c_{k+2} = -\frac{p(k)}{(k+2)(k+1)} c_k, \tag{2.1.2}$$

where $p(k) = \alpha\, k(k-1) + \beta\, k + \gamma, \quad k = 0, 1, 2, \ldots$. *Moreover, the coefficients of the even and odd powers of* x *can be computed separately as*

$$c_{2m+2} = -\frac{p(2m)}{(2m+2)(2m+1)} c_{2m}, \quad c_{2m+3} = -\frac{p(2m+1)}{(2m+3)(2m+2)} a_{2m+1}, \quad m \geq 0, \tag{2.1.3}$$

with a general solution

$$y = c_0 \left(1 + \sum_{m=1}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} p(2j)\right] \frac{x^{2m}}{(2m)!}\right) + c_1 \left(x + \sum_{m=1}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} p(2j+1)\right] \frac{x^{2m+1}}{(2m+1)!}\right). \tag{2.1.4}$$

In terms of the hypergeometric series, the general solution reads

$$\begin{aligned} y = {} & c_0\, {}_2F_1\left(\frac{-\alpha+\beta-\sqrt{(\alpha-\beta)^2 - 4\,\alpha\,\gamma}}{4\alpha}, \frac{-\alpha+\beta+\sqrt{(\alpha-\beta)^2 - 4\,\alpha\,\gamma}}{4\alpha}; \frac{1}{2}; -\alpha\, x^2\right) \\ & + c_1\ x\, {}_2F_1\left(\frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2 - 4\,\alpha\,\gamma}}{4\alpha}, \frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2 - 4\,\alpha\,\gamma}}{4\alpha}; \frac{3}{2}; -\alpha\, x^2\right). \end{aligned} \tag{2.1.5}$$

Proof. Since $x = 0$ is an ordinary point of the differential equation, the general solution has the series form

$$y = \sum_{k=0}^{\infty} c_k x^k \quad \Longrightarrow \quad y' = \sum_{k=1}^{\infty} k\, c_k\, x^{k-1}, \quad y'' = \sum_{k=2}^{\infty} k\,(k-1)\, c_k x^{k-2} \tag{2.1.6}$$

Substituting by y, y' and y'' into the differential equation yields

$$(1+\alpha\,x^2)\,\sum_{k=2}^{\infty}k\,(k-1)\,c_k x^{k-2}+\beta\,x\,\sum_{k=1}^{\infty}k\,c_k\,x^{k-1}+\gamma\,\sum_{k=0}^{\infty}c_k x^k=0,$$

whence

$$\sum_{k=2}^{\infty}k\,(k-1)\,c_k x^{k-2}+\sum_{k=2}^{\infty}\alpha k\,(k-1)\,c_k x^k+\sum_{k=1}^{\infty}\beta k\,c_k\,x^k+\sum_{k=0}^{\infty}\gamma\,c_k x^k=0, \tag{2.1.7}$$

Shifting the first sum allows for

$$\sum_{k=0}^{\infty}(k+2)\,(k+1)\,c_{k+2}x^k+\sum_{k=2}^{\infty}\alpha k\,(k-1)\,c_k x^k+\sum_{k=1}^{\infty}\beta k\,c_k\,x^k+\sum_{k=0}^{\infty}\gamma\,c_k x^k=0. \tag{2.1.8}$$

To unify the indices, we break down the sums as

$$\begin{aligned}2c_2+6c_3x+\sum_{k=2}^{\infty}(k+2)\,(k+1)\,c_{k+2}x^k+\sum_{k=2}^{\infty}\alpha k\,(k-1)\,c_k x^k&\\+\beta c_1 x+\sum_{k=2}^{\infty}\beta k\,c_k\,x^k+\gamma\,c_0+\gamma\,c_1\,x+\sum_{k=2}^{\infty}\gamma\,c_k x^k&=0.\end{aligned}$$

Thus,

$$\begin{aligned}&(2c_2+\gamma\,c_0)+(6c_3+(\beta+\gamma)c_1)\,x\\&\quad+\sum_{k=2}^{\infty}\left[(k+2)\,(k+1)\,c_{k+2}+(k\,(k-1)\alpha+k\,\beta+\gamma)\,c_k\right]\,x^k=0.\end{aligned} \tag{2.1.9}$$

Comparing the coefficient of x and using the fact that the set of functions $\{1,x,x^2,\dots\}$ is linearly independent set, we have

$$c_2=-\frac{\gamma}{2}\,c_0,\quad c_3=-\frac{(\beta+\gamma)c_1}{6},\quad c_{k+2}=-\frac{k\,(k-1)\alpha+k\,\beta+\gamma}{(k+2)(k+1)}\,c_k,\quad k\geq 0. \tag{2.1.10}$$

The recurrence relation can be written as

$$c_{k+2}=-\frac{p(k)}{(k+2)(k+1)}\,c_k,\qquad p(k)=k\,(k-1)\alpha+k\,\beta+\gamma,\qquad k\geq 0. \tag{2.1.11}$$

The coefficients of the even and odd powers of x can be computed separately as

$$(Even)\qquad k=2m\Longrightarrow c_{2m+2}=-\frac{p(2m)}{(2m+2)(2m+1)}c_{2m},\quad m\geq 0 \tag{2.1.12}$$

$$(Odd)\qquad k=2m+1\Longrightarrow c_{2m+3}=-\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1},\quad m\geq 0. \tag{2.1.13}$$

For the even coefficients, we note

$$c_{2m}=(-1)^m\frac{p(2m-2)}{(2m)(2m-1)}\times\frac{p(2m-4)}{(2m-2)(2m-3)}\times\cdots\times\frac{p(0)}{(2)(1)}c_0$$

$$= (-1)^m \frac{p(2m-2)(2m-4)\ldots p(0)}{(2m)!} = \frac{(-1)^m}{(2m)!}\prod_{j=0}^{m-1} p(2j), \tag{2.1.14}$$

while for the odd coefficients

$$\begin{aligned} c_{2m+1} &= (-1)^m \frac{p(2m-1)}{(2m+1)(2m)} \times \frac{p(2m-3)}{(2m-1)(2m-2)} \times \cdots \times \frac{p(1)}{(3)(2)} \\ &= (-1)^m \frac{p(2m-1)p(2m-3)\ldots p(1)}{(2m+1)!} = \frac{(-1)^m}{(2m+1)!}\prod_{j=0}^{m-1} p(2j+1). \end{aligned} \tag{2.1.15}$$

Hence, the general solution is

$$y = c_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} p(2j)\right] \frac{x^{2m}}{(2m)!} + c_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} p(2j+1)\right] \frac{x^{2m+1}}{(2m+1)!}, \tag{2.1.16}$$

where

$$p(2j) = 2j(2j-1)\alpha + 2j\beta + \gamma \quad \textit{and} \quad p(2j+1) = 2j(2j+1)\alpha + (2j+1)\beta + \gamma. \tag{2.1.17}$$

Since

$$p(2j) = 4\alpha\left(j - \frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)\left(j - \frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)$$

and

$$p(2j+1) = 4\alpha\left(j + \frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)\left(j + \frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)$$

we have

$$\prod_{j=0}^{m-1} p(2j) = 4^m\alpha^m \left(-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)_m \left(-\frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)_m, \tag{2.1.18}$$

and

$$\prod_{j=0}^{m-1} p(2j+1) = 4^m\alpha^m \left(\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)_m \left(\frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)_m. \tag{2.1.19}$$

Further, using

$$(2m)! = 4^m \left(\frac{1}{2}\right)_m m!, \qquad (2m+1)! = 4^m \left(\frac{3}{2}\right)_m m!, \tag{2.1.20}$$

the general solution reads

$$y_m = c_0 \sum_{m=0}^{\infty} \frac{\left(-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)_m \left(-\frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)_m}{\left(\frac{1}{2}\right)_m} \frac{(-\alpha x^2)^m}{m!}$$

$$+\,c_1 x\sum_{m=0}^{\infty}\frac{\left(\dfrac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)_m\left(\dfrac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}\right)_m}{\left(\dfrac{3}{2}\right)_m}\frac{(-\alpha x^2)^m}{m!},\tag{2.1.21}$$

which gives the hypergeometric series representation required. □

2.2 Polynomial solutions

Theorem 2.2.1. *An n-degree polynomial solution of the differential equation*

$$(1+\alpha\,x^2)\,y''+\beta\,x\,y'+\gamma\,y=0\tag{2.2.1}$$

occurs if and only if

$$\gamma=-n\,(n-1)\,\alpha-n\,\beta\,,\quad n=0,1,2,\dots.$$

In this case, the general solution is given by

$$y_n=c_0\,{}_2F_1\left(-\frac{n}{2},\frac{n}{2}+\frac{\beta}{2\alpha}-\frac{1}{2};\frac{1}{2};-\alpha\,x^2\right)+c_1\;x\,{}_2F_1\left(\frac{1-n}{2},\frac{n}{2}+\frac{\beta}{2\alpha};\frac{3}{2};-\alpha\,x^2\right).\tag{2.2.2}$$

If $n=2m, m=0,1,2,\dots$ is an even non-negative integer, the even-degree polynomials of the differential equation

$$(1+\alpha\,x^2)\,y''+\beta\,x\,y'-2m\,((2m-1)\,\alpha+\beta)\,y=0\tag{2.2.3}$$

are given by

$$y_m(x)={}_2F_1\left(-m,m+\frac{\beta}{2\alpha}-\frac{1}{2};\frac{1}{2};-\alpha\,x^2\right),\quad m=0,1,2,\dots,\tag{2.2.4}$$

while if $n=2m+1$ an odd non-negative integer, the odd polynomial solutions of

$$(1+\alpha\,x^2)\,y''+\beta\,x\,y'-(2m+1)(2m\,\alpha+\beta)\,y=0\tag{2.2.5}$$

are

$$y_m(x)=x\,{}_2F_1\left(-m,m+\frac{\beta}{2\alpha}+\frac{1}{2};\frac{3}{2};-\alpha\,x^2\right),\quad m=0,1,2,\dots.\tag{2.2.6}$$

The fundamental solutions consist of two polynomial solutions of degrees r and s respectively, if

$$\beta=-2\alpha(r+s),\quad \text{and}\quad \gamma=2\alpha r(1+2s).\tag{2.2.7}$$

in which case, the polynomial solutions occurs if

$$\alpha\,n(n-1)+\beta\,n+\gamma=\alpha(n-2r)(n-2s-1),\tag{2.2.8}$$

where r and s are non-negative integers. The polynomial solutions of the differential equation

$$(1+\alpha\,x^2)\,y''-2\alpha\,(r+s)\,x\,y'+2\,\alpha\,r\,(1+2s)\,y=0\tag{2.2.9}$$

are given by

$$y(x)=c_0\,{}_2F_1\left(-r,-s-\frac{1}{2};\frac{1}{2};-\alpha x^2\right)+c_1\,x\,{}_2F_1\left(-s,-r+\frac{1}{2};\frac{3}{2};-\alpha x^2\right).\tag{2.2.10}$$

Proof. The recurrence relation

$$c_{k+2}=-\frac{p(k)}{(k+2)(k+1)}\,c_k,\qquad p(k)=k\,(k-1)\alpha+k\,\beta+\gamma,\qquad k\geq 0.$$

terminates if and only if for some $n\geq 0$, we have

$$p(n)=0\Longrightarrow n\,(n-1)\,\alpha+n\,\beta+\gamma=0.$$

From the general solution

$$y=c_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j)\right]\frac{x^{2m}}{(2m)!}+c_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j+1)\right]\frac{x^{2m+1}}{(2m+1)!}$$

we note

$$p(2j)=2j(2j-1)\alpha+2j\beta+\gamma\quad\textit{and}\quad p(2j+1)=2j(2j+1)\alpha+(2j+1)\beta+\gamma,$$

the solution of the linear system $p(2r)=0$ and $p(2s+1)=0$ for β and γ yields

$$\beta=-2\alpha(r+s),\quad\textit{and}\quad\gamma=2\alpha r(1+2s).$$

whence,

$$\alpha\,n(n-1)+\beta\,n+\gamma=\alpha(n-2r)(n-2s-1),\qquad s,r=0,1,2,\ldots$$

that yields

$$\begin{aligned}y&=c_0\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(j-r)(2j-2s-1)\right]\frac{(-2\alpha x^2)^m}{(2m)!}\\&\quad+c_1\,x\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j-2r+1)(j-s)\right]\frac{(-2\alpha x^2)^x}{(2m+1)!}.\end{aligned}$$

Since

$$\prod_{j=0}^{m-1}(j-r)=(-r)_m=0,\qquad m>r,$$

the first series terminate to a polynomial of degree r and similarly the second series also terminate for all $m>s$. In this case, we have the fundamental set of two polynomial solutions given by

$$y=c_0\sum_{m=0}^{r}\left[\prod_{j=0}^{m-1}(j-r)(2j-2s-1)\right]\frac{(-2\alpha x^2)^m}{(2m)!}+c_1x\sum_{m=0}^{s}\left[\prod_{j=0}^{m-1}(2j+1-2r)(j-s)\right]\frac{(-2\alpha x^2)^m}{(2m+1)!}.$$

The conclusion of the theorem also follows by noticing that the hypergeometric functions

$$\begin{aligned}y&=c_0\,{}_2F_1\left(\frac{-\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\,\alpha\,\gamma}}{4\alpha},\frac{-\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\,\alpha\,\gamma}}{4\alpha};\frac{1}{2};-\alpha\,x^2\right)\\&\quad+c_1\;x\,{}_2F_1\left(\frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\,\alpha\,\gamma}}{4\alpha},\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\,\alpha\,\gamma}}{4\alpha};\frac{3}{2};-\alpha\,x^2\right),\end{aligned}$$

terminates and produce of polynomial solutions of degrees r and s, respectively, if

$$\frac{-\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}=-r,\qquad\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{4\alpha}=-s,$$

which yields by solving this system for β and γ,

$$\beta=-2\,\alpha\,(r+s),\quad\textit{and}\quad\gamma=2\,\alpha\,r\,(1+2s),$$

as mentioned earlier. $\square$

2.3 Applications

2.3.1 Legendre differential equation

Lemma 2.3.1. *The coefficients $\{c_n\}$ in the infinite series solution $y(x)=\sum_{n=0}^{\infty} c_n\,x^n$ of the differential equation*

$$(1-x^2)\,y''-2\,b\,x\,y'+a\,(a+2\,b-1)\,y=0 \tag{2.3.1}$$

satisfy the two-term recurrence relation

$$c_{k+2}=\frac{(k-a)(a+2b+k-1).}{(k+2)(k+1)}c_k, \quad k=0,1,2,\ldots. \tag{2.3.2}$$

Moreover, the coefficients of the even and odd powers of x can be computed separately as

$$\begin{aligned} c_{2m+2}&=\frac{(2m-a)(a+2b+2m-1)}{(2m+2)(2m+1)}c_{2m}, \quad m\geq 0,\\ c_{2m+3}&=\frac{(2m-a+1)(a+2b+2m)}{(2m+3)(2m+2)}a_{2m+1}, \quad m\geq 0. \end{aligned} \tag{2.3.3}$$

where c_0 and c_1 are arbitrary. Hence, the general solution is

$$\begin{aligned} y&=c_0\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j-a)(a+2b+2j-1)\right]\frac{x^{2m}}{(2m)!}\\ &+c_1\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j-a+1)(a+2b+2j)\right]\frac{x^{2m+1}}{(2m+1)!} \end{aligned} \tag{2.3.4}$$

that can be written as

$$y=c_0\,{}_2F_1\left(\frac{a+2b-1}{2},-\frac{a}{2};\frac{1}{2};\,x^2\right)+c_1\;x\,{}_2F_1\left(\frac{a+2b}{2},\frac{1-a}{2};\frac{3}{2};x^2\right). \tag{2.3.5}$$

Proof. The proof of this lemma follow using Theorem 2.1.1 with $\alpha=-1$, $\beta=-2b$ and $\gamma=a(a+2b-1)$. □

Corollary. *The second-order linear differential equation*

$$(1-\,x^2)\,y''+2(r+s)\,x\,y'-2r(1+2s)\,y=0 \tag{2.3.6}$$

where r and s are non-negative integers, has the polynomial solutions

$$y=c_0\,{}_2F_1\left(-r,-s-\frac{1}{2};\frac{1}{2};x^2\right)+c_1\,x\,{}_2F_1\left(-s,-r+\frac{1}{2};\frac{3}{2};x^2\right).$$

Proof. Solve $(a+2b-1)/2=-r$ and $(a+2b)/2=-s$ for a and b and substitute in (2.3.5). □

2.3.2 Hermite differential equation

Hermite differential equation

$$y''+\beta\,x\,y'+\gamma\,y=0 \tag{2.3.7}$$

follows from (2.1.1) through the limit operation as α approach to zero. The coefficients $\{c_k\}_{k=0}^{\infty}$ of the infinite series solution $y(x)=\sum_{k=0}^{\infty}c_k\,x^k$ of the differential equation (2.1.2) satisfy the two-term recurrence relation

$$c_{k+2}=-\frac{\beta\,k+\gamma}{(k+2)(k+1)}c_k,\quad k=0,1,2,\dots. \tag{2.3.8}$$

Moreover, the coefficients of the even and odd powers of x can be computed separately as

$$c_{2m+2}=-\frac{2m\beta+\gamma}{(2m+2)(2m+1)}c_{2m},\quad c_{2m+3}=-\frac{(2m+1)\beta+\gamma}{(2m+3)(2m+2)}c_{2m+1},\quad m\geq 0, \tag{2.3.9}$$

with a general solution

$$y=c_0\left(1+\sum_{m=1}^{\infty}\frac{\left(\frac{\gamma}{2\beta}\right)_m}{\left(\frac{1}{2}\right)_m}\frac{\left(-\frac{\beta x^2}{2}\right)^m}{m!}\right)+c_1\,x\,\left(1+\sum_{m=1}^{\infty}\frac{\left(\frac{1}{2}+\frac{\gamma}{2\beta}\right)_m}{\left(\frac{3}{2}\right)_m}\frac{\left(-\frac{\beta x^2}{2}\right)^m}{m!}\right). \tag{2.3.10}$$

In terms of the hypergeometric series, the general solution reads

$$y=c_0\,{}_1F_1\left(\frac{\gamma}{2\beta};\frac{1}{2};-\frac{\beta\,x^2}{2}\right)+c_1\;x\;{}_1F_1\left(\frac{1}{2}+\frac{\gamma}{2\beta};\frac{3}{2};-\frac{\beta\,x^2}{2}\right). \tag{2.3.11}$$

The polynomial solutions occur when

$$\gamma=-2\,n\,\beta,\quad n=0,1,2,\dots, \tag{2.3.12}$$

with polynomial solutions of the differential equation

$$y''+\beta\,x\,y'-2\,n\,\beta\,y=0 \tag{2.3.13}$$

is given by

$$y={}_1F_1\left(-n;\frac{1}{2};-\frac{\beta}{2}\,x^2\right),\quad n=0,1,2,\dots. \tag{2.3.14}$$

while the polynomial solutions of the differential equation

$$y''+\beta\,x\,y'-(2\,n+1)\,\beta\,y=0 \tag{2.3.15}$$

are

$$y=\;x\;{}_1F_1\left(-n;\frac{3}{2};-\frac{\beta}{2}\,x^2\right),\quad n=0,1,2,\dots. \tag{2.3.16}$$

2.3.3 Constant coefficient differential equation

The general solution of the differential equation

$$y''+\gamma\,y=0,\quad \gamma>0 \tag{2.3.17}$$

is given by the limit the confluent hypergeometric function ${}_1F_1$ as β approach zero

$$\lim_{\beta\to 0}{}_1F_1\left(\frac{\gamma}{2\beta};\frac{1}{2};-\frac{\beta\,x^2}{2}\right)={}_0F_1\left(-;\frac{1}{2};-\frac{\gamma\,x^2}{4}\right)=\cos(\sqrt{\gamma}\,x), \tag{2.3.18}$$

and

$$\lim_{\beta\to 0}x\,{}_1F_1\left(\frac{1}{2}+\frac{\gamma}{2\beta};\frac{3}{2};-\frac{\beta\,x^2}{2}\right)=x\,{}_0F_1\left(-;\frac{3}{2};-\frac{\gamma\,x^2}{4}\right)=\sin(\sqrt{\gamma}\,x). \tag{2.3.19}$$

For the differential equation

$$y''+\gamma\,y=0,\quad \gamma=-|\gamma|<0 \tag{2.3.20}$$

is given by the limit the confluent hypergeometric function ${}_1F_1$ as β approach zero

$$\lim_{\beta\to 0}{}_1F_1\left(\frac{-|\gamma|}{2\beta};\frac{1}{2};-\frac{\beta\,x^2}{2}\right)={}_0F_1\left(-;\frac{1}{2};\frac{|\gamma|\,x^2}{4}\right)=\cosh(\sqrt{|\gamma|}\,x), \tag{2.3.21}$$

and

$$\lim_{\beta\to 0}x\,{}_1F_1\left(\frac{1}{2}-\frac{|\gamma|}{2\beta};\frac{3}{2};-\frac{\beta\,x^2}{2}\right)=x\,{}_0F_1\left(-;\frac{3}{2};\frac{|\gamma|\,x^2}{4}\right)=\sinh(\sqrt{|\gamma|}\,x). \tag{2.3.22}$$

Chapter 3

On the solutions of the differential equation: $(1+\alpha\, x^3)\, y'' + \beta\, x^2\, y' + \gamma\, x\, y = 0$

3.1 Series solutions

Theorem 3.1.1. *The coefficients* $\{c_k\}$ *of the infinite series solution* $y(x) = \sum_{k=0}^{\infty} c_k\, x^k$ *of the differential equation*

$$(1+\alpha\, x^3)\, y'' + \beta\, x^2\, y' + \gamma\, x\, y = 0 \tag{3.1.1}$$

satisfy

$$c_2 = c_5 = c_8 = c_{11} = \cdots = 0$$

and the remaining coefficients computed using the two-term recurrence relation

$$c_{k+3} = -\frac{p(k)}{(k+2)(k+3)}\, c_k, \quad \textit{where} \quad p(k) = \alpha k(k-1) + \beta\, k + \gamma, \qquad k = 0, 1, 2, \ldots. \tag{3.1.2}$$

Moreover, the coefficients of the even and odd powers of x *can be computed separately as*

$$c_{3m+3} = -\frac{p(3m)}{(3m+2)(3m+3)} c_{3m}, \quad c_{4m+4} = -\frac{p(3m+4)}{(3m+3)(3m+4)} c_{2m+1}, \quad m \geq 0, \tag{3.1.3}$$

and the general solution is

$$y = c_0 \sum_{m=0}^{\infty} (-1)^m \left(\prod_{j=0}^{m-1} \frac{p(3j)}{(3j+2)} \right) \frac{x^{3m}}{3^m\, m!} + c_1 \sum_{m=0}^{\infty} (-1)^m \left(\prod_{j=0}^{m-1} \frac{p(3j+1)}{(3j+4)} \right) \frac{x^{3m+1}}{3^m\, m!}, \tag{3.1.4}$$

where c_0 *and* c_1 *are arbitrarily constants.*

Proof. The point $x = 0$ is an ordinary point, hence the general solution takes the form

$$y = \sum_{k=0}^{\infty} c_k x^k$$

where c_0 and c_1 are arbitrary constants. The general solution has the series form

$$y = \sum_{k=0}^{\infty} c_k x^k \quad \Longrightarrow \quad y' = \sum_{k=1}^{\infty} k\, c_k\, x^{k-1}, \quad y'' = \sum_{k=2}^{\infty} k\,(k-1)\, c_k x^{k-2} \tag{3.1.5}$$

Substituting by y, y' and y'' into the differential equation yields

$$(1+\alpha\, x^3) \sum_{k=2}^{\infty} k\,(k-1)\, c_k x^{k-2} + \beta\, x^2 \sum_{k=1}^{\infty} k\, c_k\, x^{k-1} + \gamma\, x \sum_{k=0}^{\infty} c_k x^k = 0,$$

whence

$$\sum_{k=2}^{\infty} k\,(k-1)\, c_k x^{k-2} + \sum_{k=2}^{\infty} \alpha k\,(k-1)\, c_k x^{k+1} + \sum_{k=1}^{\infty} \beta k\, c_k\, x^{k+1} + \sum_{k=0}^{\infty} \gamma\, c_k x^{k+1} = 0.$$

Shifting the first sum allow for writing

$$\sum_{k=0}^{\infty}(k+2)\,(k+1)\, c_{k+2} x^k + \sum_{k=3}^{\infty} \alpha(k-1)\,(k-2)\, c_{k-1} x^k + \sum_{k=2}^{\infty} \beta(k-1)\, c_{k-1}\, x^k + \sum_{k=1}^{\infty} \gamma\, c_{k-1} x^k = 0.$$

To unify the indices, we break down the sums as

$$\begin{aligned} 2c_2 + 6c_3 x + 12 c_4 x^2 + \sum_{k=3}^{\infty}(k+2)\,(k+1)\, c_{k+2} x^k + \sum_{k=3}^{\infty} \alpha(k-1)\,(k-2)\, c_{k-1} x^k \\ + \beta c_1 x^2 + \sum_{k=3}^{\infty} \beta(k-1)\, c_{k-1}\, x^k + \gamma\, c_0\, x + \gamma c_1 x^2 + \sum_{k=3}^{\infty} \gamma\, c_{k-1} x^k = 0. \end{aligned}$$

Thus,

$$\begin{aligned} & 2c_2 + (6c_3 + \gamma c_0)\, x + (12 c_4 + \beta c_1 + \gamma c_1) x^2 \\ & \quad + \sum_{k=3}^{\infty} \left[(k+2)\,(k+1)\, c_{k+2} + ((k-1)\,(k-2)\alpha + (k-1)\,\beta + \gamma)\, c_{k-1}\right]\, x^k = 0. \end{aligned}$$

Comparing the coefficient of x, we have

$$\begin{gathered} c_2 = 0, \quad c_3 = -\frac{\gamma}{6} c_0, \quad c_4 = -\frac{\beta+\gamma}{12}\, c_1, \\ c_{k+2} = -\frac{((k-1)\,(k-2)\alpha + (k-1)\,\beta + \gamma)}{(k+1)(k+2)}\, c_{k-1}, \qquad k = 1, 2, \ldots. \end{gathered}$$

The recurrence relation can be written as

$$c_{k+3} = -\frac{k\,(k-1)\alpha + k\,\beta + \gamma}{(k+2)(k+3)}\, c_k = -\frac{p(k)}{(k+2)(k+3)}\, c_k, \qquad k = 0, 1, 2, \ldots$$

where

$$p(k) = k\,(k-1)\alpha + k\,\beta + \gamma.$$

Thus

$$\begin{gathered} c_2 = 0, \quad c_3 = -\frac{p(0)}{2\cdot 3}\, c_0, \quad c_4 = -\frac{p(1)}{3\cdot 4}\, c_1, \\ c_5 = -\frac{p(2)}{4\cdot 5}\, c_2 = 0, \quad c_6 = -\frac{p(3)}{5\cdot 6}\, c_3 = (-1)^2 \frac{p(3)}{5\cdot 6}\, \frac{p(0)}{2\cdot 3}\, c_0 \\ c_7 = -\frac{p(4)}{6\cdot 7}\, c_4 = (-1)^2 \frac{p(4)}{6\cdot 7}\, \frac{p(1)}{3\cdot 4}\, c_1, \quad c_8 = -\frac{p(5)}{7\cdot 8}\, c_5 = 0 \end{gathered}$$

$$c_9 = -\frac{p(6)}{8\cdot 9}\,c_6 = (-1)^3\frac{p(6)}{8\cdot 9}\,\frac{p(3)}{5\cdot 6}\,\frac{p(0)}{2\cdot 3}\,c_0,\ \ldots$$

The coefficients of the $3m, m=0,1,2,\ldots$ and $3m+1, m=0,1,2,\ldots$ powers of x can be computed separately as

$$c_{3m} = (-1)^m\prod_{j=0}^{m-1}\frac{p(3j)}{(3j+2)(3j+3)}\,c_0 = \frac{(-1)^m}{3^m m!}\prod_{j=0}^{m-1}\frac{p(3j)}{(3j+2)}\,c_0,\quad m=1,2,\ldots,$$

$$c_{3m+4} = (-1)^{m+1}\prod_{j=0}^{m-1}\frac{p(3j+1)}{(3j+3)(3j+4)}\,c_1 = \frac{(-1)^{m+1}}{3^m m!}\prod_{j=0}^{m-1}\frac{p(3j+1)}{(3j+4)}\,c_1.\quad m=0,1,2,\ldots.$$

The general solution is then

$$\begin{aligned} y &= \sum_{m=0}^{\infty} c_m\,x^m = c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+\ldots\\ &= c_0\sum_{m=0}^{\infty}(-1)^m\left(\prod_{j=0}^{m-1}\frac{p(3j)}{(3j+2)}\right)\frac{x^{3m}}{3^m\,m!} + \underbrace{c_1}_{=-c_1}\sum_{m=0}^{\infty}(-1)^m\left(\prod_{j=0}^{m-1}\frac{p(3j+1)}{(3j+4)}\right)\frac{x^{3m+1}}{3^m\,m!},\end{aligned}$$

where c_0 and c_1 are arbitrarily constants. □

Note, in terms of the Pochhammer symbol, we may write

$$\begin{aligned}\prod_{j=0}^{m-1}\frac{p(3j)}{(3j+2)} &= \frac{1}{3^m}\prod_{j=0}^{m-1}\frac{p(3j)}{\left(j+\frac{2}{3}\right)}\\ &= \frac{3^{2m}\alpha^m}{3^m\left(\frac{2}{3}\right)_m}\prod_{j=0}^{m-1}\left(j-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)\left(j-\frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)\\ &= \frac{3^m\alpha^m}{\left(\frac{2}{3}\right)_m}\left(-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)_m\left(-\frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)_m,\end{aligned}$$

and

$$\begin{aligned}\prod_{j=0}^{m-1}\frac{p(3j+1)}{(3j+4)} &= \frac{1}{3^m}\prod_{j=0}^{m-1}\frac{p(3j+1)}{\left(j+\frac{4}{3}\right)}\\ &= \frac{3^{2m}\alpha^m}{3^m\left(\frac{4}{3}\right)_m}\prod_{j=0}^{m-1}\left(j+\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)\left(j+\frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)\\ &= \frac{3^m\alpha^m}{\left(\frac{4}{3}\right)_m}\left(\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)_m\left(\frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)_m.\end{aligned}$$

The series solution can be written as

$$\begin{aligned} y &= c_0\sum_{m=0}^{\infty}\frac{\left(-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)_m\left(-\frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)_m}{\left(\frac{2}{3}\right)_m}\frac{(-\alpha x^3)^m}{m!}\\ &+ c_1 x\sum_{m=0}^{\infty}\frac{\left(\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)_m\left(\frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}\right)_m}{\left(\frac{4}{3}\right)_m}\frac{(-\alpha x^3)^m}{m!}\end{aligned}$$

$$
\begin{aligned}
&= c_0\,{}_2F_1\left(-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha},-\frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha};\frac{2}{3},-\alpha x^3\right)\\
&+c_1\,x\,{}_2F_1\left(\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha},\frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha};\frac{4}{3},-\alpha x^3\right)
\end{aligned}
$$

3.2 Polynomial solutions

Theorem 3.2.1. *The n-degree polynomial solution of the differential equation*

$$(1+\alpha\,x^3)\,y''+\beta\,x^2\,y'+\gamma\,x\,y=0 \tag{3.2.1}$$

occurs if and only if

$$\gamma=-n\,(n-1)\,\alpha-n\,\beta\,,\quad n=3m,3m+1,\quad m=0,1,2,\ldots. \tag{3.2.2}$$

In which case, the $3m$-degree polynomials of the differential equation

$$(1+\alpha\,x^3)\,y''+\beta\,x^2\,y'-3m\,((3m-1)\,\alpha+\beta)\,x\,y=0 \tag{3.2.3}$$

are given by

$$y_m={}_2F_1\left(-m,m+\frac{\beta}{3\alpha}-\frac{1}{3};\frac{2}{3};-\alpha\,x^3\right),\quad m=0,1,2,\ldots, \tag{3.2.4}$$

while the $(3m+1)$-degree polynomials of the differential equation

$$(1+\alpha\,x^3)\,y''+\beta\,x^2\,y'-(3m+1)\,(3m\,\alpha+\beta)\,x\,y=0 \tag{3.2.5}$$

are given by

$$y_m=x\,{}_2F_1\left(-m,m+\frac{\beta}{3\alpha}+\frac{1}{3};\frac{4}{3};-\alpha\,x^3\right),\quad m=0,1,2,\ldots. \tag{3.2.6}$$

Proof. From the recurrence relation

$$c_2=0,\quad c_{k+3}=-\frac{k\,(k-1)\alpha+k\,\beta+\gamma}{(k+2)(k+3)}\,c_k$$

the infinite series terminates if

$$n\,(n-1)\alpha+n\,\beta+\gamma=0\Longrightarrow\gamma=-n(n-1)\alpha-n\,\beta,\quad for\ some\ n\geq 0.$$

The polynomial solutions in the power of x^{3m} follow by putting

$$-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}=-m,\quad \text{which implies}\quad \gamma=-3m((3m-1)\alpha+\beta).$$

The polynomial solutions in the power of x^{3m+1} follows by putting

$$\frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}=-m,\quad \text{which implies}\quad \gamma=-(3m+1)(3m\alpha+\beta).$$

□

3.3 A fundamental set of two polynomial solutions

The fundamental set will consist of two polynomial solutions if

$$-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}=-r \quad \textit{and} \quad \frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{6\alpha}=-s, \tag{3.3.1}$$

for nonnegative integers r and s, that yields for β and γ, the requirements

$$\beta=-3\alpha(r+s), \qquad \gamma=3\alpha r(1+3s). \tag{3.3.2}$$

The polynomial solutions of the differential equation

$$(1+\alpha\,x^3)\,y''-3\alpha(r+s)\,x^2\,y'+3\alpha\,r\,(1+3s)\,x\,y=0 \tag{3.3.3}$$

are given, for $s,r=0,1,2,\ldots,$

$$y=c_0\,{}_2F_1\left(-r,-\frac{1+3s}{3};\frac{2}{3};-\alpha x^3\right)+c_1\,x\,{}_2F_1\left(\frac{1-3r}{3},-s;\frac{4}{3};-\alpha x^3\right). \tag{3.3.4}$$

Chapter 4

On the solutions of the differential equation: $(1+\alpha\, x^{k+2})\, y'' + \beta\, x^{k+1}\, y' + \gamma\, x^k\, y = 0$

4.1 Series solutions

Theorem 4.1.1. *The coefficients $\{c_k\}$ of the infinite series solution $y(x) = \sum_{k=0}^{\infty} c_k\, x^k$ of the differential equation*

$$(1+\alpha\, x^{k+2})\, y'' + \beta\, x^{k+1}\, y' + \gamma\, x^k\, y = 0, \qquad k \geq 0, \tag{4.1.1}$$

satisfy

$$c_\ell = 0, \qquad \text{for all} \quad 2 \leq \ell \leq k+1.$$

and the remaining coefficients computed using the recurrence relation

$$c_{\ell+k+2} = -\frac{p(\ell)}{(\ell+k+1)(\ell+k+2)}\, c_\ell, \quad p(\ell) = \ell\,(\ell-1)\alpha + \ell\,\beta + \gamma, \quad \ell = 0, 1, 2, \ldots. \tag{4.1.2}$$

Moreover, the coefficients of the $(k+2)m$ and $(k+2)m+1$ powers of x can be computed separately using

$$c_{(k+2)(m+1)} = \frac{(-1)^{m+1}}{(m+1)!(k+2)^{m+1}} \left(\prod_{j=0}^{m} \frac{p(j(k+2))}{((j+1)k+2j+1)} \right) c_0, \tag{4.1.3}$$

$$c_{(m+1)(k+2)+1} = \frac{(-1)^{m+1}}{(m+1)!(k+2)^{m+1}} \left(\prod_{j=0}^{m} \frac{p(j(k+2)+1)}{((j+1)k+2j+3)} \right) c_1, \tag{4.1.4}$$

where c_0 and c_1 are arbitrarily constants. The general solution reads in terms of the hypergeometric series

$$\begin{aligned} y = {} & c_0\, {}_2F_1\left(-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}, -\frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}; \frac{k+1}{k+2}; -\alpha\, x^{k+2} \right) \\ & + c_1 x\, {}_2F_1\left(\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}, \frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}; \frac{k+3}{k+2}; -\alpha\, x^{k+2} \right). \end{aligned} \tag{4.1.5}$$

Proof. The point $x=0$ is an ordinary point, hence the general solution takes the form

$$y=\sum_{\ell=0}^{\infty}c_\ell x^\ell$$

where c_0 and c_1 are arbitrary constants. The general solution has the series form

$$y=\sum_{\ell=0}^{\infty}c_\ell x^\ell \quad\Longrightarrow\quad y'=\sum_{\ell=1}^{\infty}\ell\,c_\ell\,x^{\ell-1},\quad y''=\sum_{\ell=2}^{\infty}\ell\,(\ell-1)\,c_\ell x^{\ell-2}$$

Substituting by y, y' and y'' into the differential equation yields

$$(1+\alpha\,x^{k+2})\sum_{\ell=2}^{\infty}\ell\,(\ell-1)\,c_\ell x^{\ell-2}+\beta\,x^{k+1}\sum_{\ell=1}^{\infty}\ell\,c_\ell\,x^{\ell-1}+\gamma\,x^k\sum_{\ell=0}^{\infty}c_\ell x^\ell=0,$$

whence

$$\sum_{\ell=2}^{\infty}\ell\,(\ell-1)\,c_\ell x^{\ell-2}+\alpha\sum_{\ell=2}^{\infty}\ell\,(\ell-1)\,c_\ell x^{\ell+k}+\beta\sum_{\ell=1}^{\infty}\ell\,c_\ell\,x^{\ell+k}+\gamma\sum_{\ell=0}^{\infty}c_\ell x^{\ell+k}=0.$$

Shifting the first sum allows for

$$\sum_{\ell=-k}^{\infty}(\ell+k+2)\,(\ell+k+1)\,c_{\ell+k+2}x^{\ell+k}+\alpha\sum_{\ell=2}^{\infty}\ell\,(\ell-1)\,c_\ell x^{\ell+k}+\beta\sum_{\ell=1}^{\infty}\ell\,c_\ell\,x^{\ell+k}+\gamma\sum_{\ell=0}^{\infty}c_\ell x^{\ell+k}=0,$$

that yields

$$\sum_{\ell=0}^{\infty}(\ell+k+2)\,(\ell+k+1)\,c_{\ell+k+2}x^{\ell+k}+\alpha\sum_{\ell=2}^{\infty}\ell\,(\ell-1)\,c_\ell x^{\ell+k}+\beta\sum_{\ell=1}^{\infty}\ell\,c_\ell\,x^{\ell+k}+\gamma\sum_{\ell=0}^{\infty}c_\ell x^{\ell+k}=0,$$

To unify the indices, we break down the sums as

$$(k+2)\,(k+1)\,c_{k+2}x^k+(k+3)\,(k+2)\,c_{k+3}x^{k+1}+\sum_{\ell=2}^{\infty}(\ell+k+2)\,(\ell+k+1)\,c_{\ell+k+2}x^{\ell+k}$$
$$+\,\alpha\sum_{\ell=2}^{\infty}\ell\,(\ell-1)\,c_\ell x^{\ell+k}+\beta c_1x^{k+1}+\beta\sum_{\ell=2}^{\infty}\ell\,c_\ell\,x^{\ell+k}+\gamma c_0x^k+\gamma c_1x^{k+1}+\gamma\sum_{\ell=2}^{\infty}c_\ell x^{\ell+k}=0.$$

Comparing the coefficient of x, we have for $\ell=1,2,\ldots$

$$c_{k+2}=-\frac{\gamma}{(k+2)(k+1)}c_0,\quad c_{k+3}=-\frac{\beta+\gamma}{(k+2)(k+3)}c_1,\quad c_{\ell+k+2}=-\frac{\ell\,(\ell-1)\alpha+\ell\,\beta+\gamma}{(\ell+k+1)(\ell+k+2)}\,c_\ell.$$

The recurrence relation can be written as

$$c_{\ell+k+2}=-\frac{p(\ell)}{(\ell+k+1)(\ell+k+2)}\,c_\ell,\quad p(\ell)=\ell\,(\ell-1)\alpha+\ell\,\beta+\gamma,\quad \ell=0,1,2,\ldots.$$

Clearly,

$$c_{k+4}=-\frac{p(2)}{(k+3)(k+4)}\,c_2,\quad c_{k+5}=-\frac{p(3)}{(k+4)(k+5)}\,c_3,\quad\ldots,\quad c_{2k+3}=-\frac{p(k+1)}{(2k+2)(2k+3)}\,c_{k+1}$$

Because none of the c_ℓ for $2 \le \ell \le k+1$ can be evaluated in terms of c_0 and c_1, it is necessary that

$$c_\ell = 0, \qquad \text{for all} \quad 2 \le \ell \le k+1.$$

The other coefficients of the series solutions can be evaluated using

$$\begin{aligned}
c_{2k+4} &= -\frac{p(k+2)}{(2k+3)(2k+4)}\, c_{k+2} = (-1)^2 \frac{p(k+2)}{(2k+3)(2k+4)}\, \frac{p(0)}{(k+2)(k+1)} c_0 \\
&= \frac{(-1)^2}{2(k+2)^2} \frac{p(k+2)}{(2k+3)} \frac{p(0)}{(k+1)} c_0, \\
c_{2k+5} &= -\frac{p(k+3)}{(2k+4)(2k+5)}\, c_{k+3} = (-1)^2 \frac{p((k+2)+1)}{(2k+4)(2k+5)}\, \frac{p(1)}{(k+2)(k+3)} c_1 \\
&= \frac{(-1)^2}{2(k+2)^2} \frac{p((k+2)+1)}{(2(k+2)+1)} \frac{p(1)}{((k+2)+1)} c_1, \\
c_{2k+6} &= -\frac{p(k+4)}{(2k+5)(2k+6)}\, c_{k+4} = 0, \quad \ldots, \quad c_{3k+5} = -\frac{p(2k+3)}{(3k+4)(2k+6)}\, c_{2k+3} = 0, \\
c_{3k+6} &= -\frac{p(2k+4)}{(3k+5)(3k+6)}\, c_{2k+4} = (-1)^3 \frac{p(2k+4)}{(3k+5)(3k+6)} \frac{p(k+2)}{(2k+3)(2k+4)} \frac{p(0)}{(k+2)(k+1)} c_0, \\
&= \frac{(-1)^3}{3!(k+2)^3} \left(\prod_{j=0}^{2} \frac{p(j(k+2))}{((j+1)k+2j+1)} \right) c_0, \\
c_{3k+7} &= -\frac{p(2k+5)}{(3k+6)(3k+7)}\, c_{2k+5} = (-1)^3 \frac{p(2(k+2)+1)}{(3k+6)(3k+7)} \frac{p((k+2)+1)}{(2k+4)(2k+5)} \frac{p(1)}{(k+2)(k+3)} c_1 \\
&= \frac{(-1)^3}{3!(k+2)^3} \left(\prod_{j=0}^{2} \frac{p(j(k+2)+1)}{((j+1)k+2j+3)} \right) c_1, \\
c_{3k+8} &= -\frac{p(2k+6)}{(3k+7)(3k+8)}\, c_{2k+6}, \quad \ldots, \quad c_{4k+7} = -\frac{p(3k+5)}{(4k+6)(4k+7)}\, c_{3k+5} = 0, \\
c_{4k+8} &= -\frac{p(3k+6)}{(4k+7)(4k+8)}\, c_{3k+6} \\
&= (-1)^4 \frac{p(3k+6)}{(4k+7)(4k+8)} \frac{p(2k+4)}{(3k+5)(3k+6)} \frac{p(k+2)}{(2k+3)(2k+4)} \frac{p(0)}{(k+2)(k+1)} c_0 \\
&= \frac{(-1)^4}{4!(k+2)^4} \left(\prod_{j=0}^{3} \frac{p(j(k+2))}{((j+1)k+2j+1)} \right) c_0, \\
c_{4k+9} &= \frac{(-1)^4}{4!(k+2)^4} \left(\prod_{j=0}^{3} \frac{p(j(k+2)+1)}{((j+1)k+2j+3)} \right) c_1, \ldots .
\end{aligned}$$

In general

$$c_{(k+2)(m+1)} = \frac{(-1)^{m+1}}{(m+1)!(k+2)^{m+1}} \left(\prod_{j=0}^{m} \frac{p(j(k+2))}{((j+1)k+2j+1)} \right) c_0,$$

$$c_{(m+1)(k+2)+1} = \frac{(-1)^{m+1}}{(m+1)!(k+2)^{m+1}} \left(\prod_{j=0}^{m} \frac{p(j(k+2)+1)}{((j+1)k+2j+3)} \right) c_1,$$

with the general solution

$$y = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(k+2)^m} \left(\prod_{j=0}^{m-1} \frac{p(j(k+2))}{((j+1)k+2j+1)} \right) x^{(k+2)m}$$

$$+ c_1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(k+2)^m} \left(\prod_{j=0}^{m-1} \frac{p(j(k+2)+1)}{((j+1)k+2j+3)} \right) x^{m(k+2)+1}.$$

Writing

$$(j+1)k+2j+1 = (k+2)\left(j+\frac{k+1}{k+2}\right), \quad (j+1)k+2j+3 = (k+2)\left(j+\frac{k+3}{k+2}\right),$$

we have

$$\prod_{j=0}^{m-1}((j+1)k+2j+1) = (k+2)^m \left(\frac{k+1}{k+2}\right)_m, \quad \prod_{j=0}^{m-1}((j+1)k+2j+3) = (k+2)^m \left(\frac{k+3}{k+2}\right)_m.$$

Further, writing

$$\begin{aligned} p(j(k+2)) &= (j(k+2))(j(k+2)-1)\alpha + (j(k+2))\beta + \gamma \\ &= \alpha(2+k)^2 \left(j + \frac{-\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}\right)\left(j - \frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}\right), \end{aligned}$$

we have

$$\prod_{j=0}^{m-1} p(j(k+2)) = \alpha^m (2+k)^{2m} \left(\frac{-\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}\right)_m \left(-\frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}\right)_m.$$

The general solution then reads

$$\begin{aligned} y &= c_0 \sum_{m=0}^{\infty} \frac{(-\alpha x^{k+2})^m}{m!} \frac{\left(-\frac{\alpha-\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{\alpha(2+k)}\right)_m \left(-\frac{\alpha-\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{\alpha(2+k)}\right)_m}{\left(\frac{k+1}{k+2}\right)_m} \\ &+ c_1 x \sum_{m=0}^{\infty} \frac{(-\alpha x^{k+2})^m}{m!} \frac{\left(\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}\right)_m \left(\frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}\right)_m}{\left(\frac{k+3}{k+2}\right)_m} \\ &= c_0\, {}_2F_1\left(\frac{\beta-\alpha+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}, \frac{\beta-\alpha-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}; \frac{k+1}{k+2}; -\alpha\, x^{k+2}\right) \\ &+ c_1\, x\, {}_2F_1\left(\frac{\alpha+\beta+\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}, \frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2-4\alpha\gamma}}{2\alpha(2+k)}; \frac{k+3}{k+2}; -\alpha\, x^{k+2}\right). \end{aligned}$$

□

4.2 Polynomial solutions

Theorem 4.2.1. *The n-degree polynomial solution of the differential equation*

$$(1+\alpha\, x^{k+2})\, y'' + \beta\, x^{k+1}\, y' + \gamma\, x^k\, y = 0, \qquad k = 0,1,2,\ldots \tag{4.2.1}$$

occurs if and only if

$$\gamma = -n\,(n-1)\,\alpha - n\,\beta\,, \quad n = (k+2)m,\ (k+2)m+1, \quad m = 0,1,2,\ldots. \tag{4.2.2}$$

The even polynomial solutions, $n = (k+2)m$, of the differential equation

$$(1+\alpha\, x^{k+2})\, y'' + \beta\, x^{k+1}\, y' - [((k+2)m)((k+2)m-1)\alpha + (k+2)m\beta]\, x^k\, y = 0, \tag{4.2.3}$$

are given by

$$y_{(k+2)m}(x) = {}_2F_1\left(-m, -\frac{\alpha-\beta}{\alpha(2+k)} + m; \frac{k+1}{k+2}; -\alpha\, x^{k+2}\right), \tag{4.2.4}$$

while the odd polynomial solutions, $n = (k+2)m+1$, of the differential equation

$$(1+\alpha\, x^{k+2})\, y'' + \beta\, x^{k+1}\, y' - [((k+2)m+1)((k+2)m)\alpha + ((k+2)m+1)\beta]\, x^k\, y = 0, \tag{4.2.5}$$

are given by

$$y_{(k+2)m+1}(x) = x\, {}_2F_1\left(-m, \frac{\alpha+\beta}{\alpha(2+k)} + m; \frac{k+3}{k+2}; -\alpha\, x^{k+2}\right). \tag{4.2.6}$$

Chapter 5

On the solutions of the differential equation: $(1+\alpha\,x+\beta\,x^2)\,y''+(\gamma+\delta\,x)\,y'+\varepsilon\,y=0$

5.1 Series solutions

Theorem 5.1.1. *The coefficients* $\{a_n\}_{n=0}^{\infty}$ *of the series solution*

$$y(x)=\sum_{n=0}^{\infty} a_k\,x^k$$

of the differential equation

$$(1+\alpha x+\beta x^2)\,y''(x)+(\gamma+\delta x)\,y'(x)+\varepsilon\,y(x)=0 \tag{5.1.1}$$

where $\alpha,\beta,\gamma,\delta,$ *and* ε *are real constants, satisfy a three-term recurrence relation*

$$a_{n+2}+\frac{\gamma+\alpha\,n}{n+2}\,a_{n+1}+\frac{\beta\,n\,(n-1)+\delta\,n+\varepsilon}{(n+2)(n+1)}\,a_n=0,\qquad n=0,1,\dots, \tag{5.1.2}$$

where the constants a_0 *and* a_1 *are chosen arbitrary.*

Proof. In the neighbourhood of the ordinary point $x=0$, the series solution take the form

$$y(x)=\sum_{n=0}^{\infty} a_n\,x^n \quad thus \quad y'(x)=\sum_{n=1}^{\infty} n\,a_n\,x^{n-1} \quad and \quad y''(x)=\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2}.$$

The substitution of $y(x), y'(x)$ and $y''(x)$ in the given differential equation yields

$$(1+\alpha x+\beta x^2)\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}+\ (\gamma+\delta x)\sum_{n=1}^{\infty} na_nx^{n-1}+\varepsilon\sum_{n=0}^{\infty} a_nx^n=0.$$

A further simplification yields

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}+\alpha\sum_{n=2}^{\infty} n(n-1)a_nx^{n-1}+\beta\sum_{n=2}^{\infty} n(n-1)a_nx^{n}+\gamma\sum_{n=1}^{\infty} na_nx^{n-1}$$
$$+\,\delta\sum_{n=1}^{\infty} na_nx^{n}+\varepsilon\sum_{n=0}^{\infty} a_nx^n=0.$$

Shifting the indices implies

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\alpha\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n+\beta\sum_{n=2}^{\infty}n(n-1)a_nx^n$$
$$+\gamma\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\delta\sum_{n=1}^{\infty}na_nx^n+\varepsilon\sum_{n=0}^{\infty}a_nx^n=0.$$

To unify the indices, we write

$$(2a_2+\gamma a_1+\varepsilon a_0)+(6a_3+2\alpha a_2+2\gamma a_2+\delta a_1+\varepsilon a_1)x$$
$$+\sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2}+\alpha n(n+1)a_{n+1}+\beta n(n-1)a_n+\gamma(n+1)a_{n+1}+\delta na_n+\varepsilon a_n]x^n=0$$

Comparing the coefficient of x:

$$a_2=-\frac{\gamma}{2}a_1-\frac{\varepsilon}{2}a_0,\qquad a_3=-\frac{(2\alpha+2\gamma)}{3}a_2-\frac{\delta+\varepsilon}{6}a_1,$$

and in general

$$a_{n+2}=-\frac{\gamma+\alpha\,n}{n+2}a_{n+1}-\frac{\beta\,n(n-1)+\delta\,n+\varepsilon}{(n+2)(n+1)}a_n,\qquad n=0,1,2,\dots,$$

where a_0 and a_1 are chosen arbitrary. □

Lemma 5.1.2. *Let r_1 and r_2 be two distinct real roots of $1+\alpha x+\beta x^2=0$. Then the exact solutions of the differential equation* (5.1.1) *can be expressed in terms of the hypergeometric functions as*

$$y(z)=C_1\,{}_2F_1\left(\frac{-\beta+\delta+\sqrt{(\beta-\delta)^2-4\beta\varepsilon}}{2\beta},\frac{-\beta+\delta-\sqrt{(\beta-\delta)^2-4\beta\varepsilon}}{2\beta};\frac{\gamma+\delta r_1}{\beta(r_1-r_2)};\frac{x-r_1}{r_2-r_1}\right)$$
$$+C_2\,z^{1+\dfrac{\gamma+\delta r_1}{\beta\,(r_2-r_1)}}\,{}_2F_1\left(\frac{\beta+\delta-\sqrt{(\beta-\delta)^2-4\beta\varepsilon}+\dfrac{2(\gamma+\delta r_1)}{r_2-r_1}}{2\beta},\right.$$
$$\left.\frac{\beta+\delta+\sqrt{(\beta-\delta)^2-4\beta\varepsilon}+\dfrac{2(\gamma+\delta)}{r_2-r_1}}{2\beta};2+\frac{\gamma+\delta r_1}{\beta(r_2-r_1)};\frac{x-r_1}{r_2-r_1}\right).\tag{5.1.3}$$

Proof. Let r_1 and r_2 be real (distinct) roots of the equation $1+\alpha\,x+\beta\,x^2=0$, that is to say,

$$r_1=\frac{-\alpha-\sqrt{\alpha^2-4\beta}}{2\beta},\quad r_2=\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}\qquad\alpha>2\sqrt{\beta}.$$

Using the change of variable

$$z=\frac{x-r_1}{r_2-r_1},\qquad\text{then}\qquad x=r_1+(r_2-r_1)z,\tag{5.1.4}$$

then

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{r_2-r_1}\frac{dy}{dz},\quad\frac{d^2y}{dx^2}=\frac{1}{(r_2-r_1)^2}\frac{d^2y}{dz^2}\tag{5.1.5}$$

and the differential equation (5.1.1) can be written as

$$\frac{1+\alpha r_1+\beta r_1^2-(\alpha+2\beta r_1)(r_1-r_2)z+\beta(r_1-r_2)^2z^2}{(r_2-r_1)^2}\frac{d^2y(z)}{dz^2}+\frac{\gamma+\delta r_1+\delta(r_2-r_1)z}{r_2-r_1}\frac{dy(z)}{dz}$$
$$+\varepsilon y(z)=0,$$

and since $1+\alpha r_1+\beta r_1^2=0$, this equation reduces to

$$\left(\beta z^2+\frac{\alpha+2\beta r_1}{r_2-r_1}\,z\right)\frac{d^2y(z)}{dz^2}+\left(\frac{\gamma+\delta r_1}{r_2-r_1}+\delta\,z\right)\frac{dy(z)}{dz}+\varepsilon\,y(z)=0, \tag{5.1.6}$$

which we may further reduce by substituting the definitions of r_1 and r_2 in the Gauss hypergeometric differential equation, to

$$z(z-1)\frac{d^2y}{dz^2}+\left(\frac{\gamma+\delta r_1}{\beta(r_2-r_1)}+\frac{\delta}{\beta}\,z\right)\frac{dy}{dz}+\frac{\varepsilon}{\beta}\,y=0. \tag{5.1.7}$$

Which has exact solutions given in terms of the hypergeometric functions as

$$y(z)=C_1{}_2F_1\left(\frac{-\beta+\delta+\sqrt{(\beta-\delta)^2-4\beta\varepsilon}}{2\beta},\frac{-\beta+\delta-\sqrt{(\beta-\delta)^2-4\beta\varepsilon}}{2\beta};\frac{\gamma+\delta r_1}{\beta(r_1-r_2)};z\right)$$
$$+C_2\,z^{1+\frac{\gamma+\delta r_1}{\beta\,(r_2-r_1)}}{}_2F_1\left(\frac{\beta+\delta-\sqrt{(\beta-\delta)^2-4\beta\varepsilon}+\frac{2(\gamma+\delta r_1)}{r_2-r_1}}{2\beta},\right.$$
$$\left.\frac{\beta+\delta+\sqrt{(\beta-\delta)^2-4\beta\varepsilon}+\frac{2(\gamma+\delta r_1)}{r_2-r_1}}{2\beta};2+\frac{\gamma+\delta r_1}{\beta(r_2-r_1)};z\right).$$

□

Because of the many important applications of these results, we summarize it as follows: *For $\alpha^2\neq 4\beta$, the general solution*

$$y(x)=C_1y_1(x)+C_2y_2(x)$$

of the differential equation

$$(1+\alpha x+\beta x^2)\,y''(x)+(\gamma+\delta x)\,y'(x)+\varepsilon\,y(x)=0 \tag{5.1.8}$$

is given in terms of the hypergeometric function as

$$y_1(x)={}_2F_1\left(-\frac{\sqrt{(\beta-\delta)^2-4\beta\varepsilon}+\beta-\delta}{2\beta},\frac{\sqrt{(\beta-\delta)^2-4\beta\varepsilon}-\beta+\delta}{2\beta};\right.$$
$$\left.\frac{\delta\left(\sqrt{\alpha^2-4\beta}+\alpha\right)-2\beta\gamma}{2\beta\sqrt{\alpha^2-4\beta}};\frac{\beta x}{\sqrt{\alpha^2-4\beta}}+\frac{\alpha}{2\sqrt{\alpha^2-4\beta}}+\frac{1}{2}\right) \tag{5.1.9}$$

5.1.1 subclass I: $(1+\alpha\, x+\beta\, x^2)\, y''+(2\,\alpha+4\,\beta x)\, y'+2\,\beta\, y=0.$

Lemma 5.1.3. *Given α and β real constants, with $\beta\neq 0$. The power series $\sum_{n=0}^{\infty} a_n x^n$ is a solution of*

$$(1+\alpha\, x+\beta\, x^2)\, y''+(2\,\alpha+4\,\beta x)\, y'+2\,\beta\, y=0 \tag{5.1.10}$$

if and only if the coefficients of the power series solution satisfy three-term recurrence relation

$$a_{n+2}+\alpha\, a_{n+1}+\beta\, a_n=0, \qquad n\geq 0. \tag{5.1.11}$$

Such an equation (5.1.11) *called a second-order homogeneous linear difference equation.*

Proof. From Theorem 5.1.1 with the values of $\gamma=2\alpha$, $\delta=4\beta$ and $\varepsilon=2\beta$, it follows

$$a_{n+2}=-\frac{\alpha(n+2)}{n+2}a_{n+1}-\frac{\beta n(n-1)+4\beta n+2\beta}{(n+2)(n+1)}a_n,$$

which simplifies to

$$a_{n+2}=-\frac{\alpha(n+2)}{n+2}a_{n+1}-\frac{\beta(n+2)(n+1)}{(n+2)(n+1)}a_n.$$

Thus,

$$a_{n+2}+\alpha a_{n+1}+\beta a_n=0, \qquad n\geq 0.$$

□

Lemma 5.1.4. *Let $p(r)=r^2+\alpha r+\beta=(r-r_1)(r-r_2)$ where r_1 and r_2 are real and distinct, if c_1 and c_2 are constants then*

$$a_n=c_1\, r_1^n+c_2\, r_2^n, \qquad n\geq 0 \tag{5.1.12}$$

and the general solution of the differential equation (5.1.10) *reads*

$$y(x)=\sum_{n=0}^{\infty}(c_1\, r_1^n+c_2\, r_2^n)\, x^n, \qquad x\in(-p,p), \tag{5.1.13}$$

where $p=\min\{1/|r_1|,1/|r_2|\}$.

Proof. The equality $p(r)=r^2+\alpha r+\beta=(r-r_1)(r-r_2)$ implies $\alpha=-(r_1+r_2)$ and $\beta=r_1r_2$. Let the coefficient of the series solutions be $a_n=c_1\, r_1^n+c_2\, r_2^n$. Then

$$\begin{aligned}
a_{n+2}&=c_1\, r_1^{n+2}+c_2\, r_2^{n+2}\\
\alpha\, a_{n+1}&=-(r_1+r_2)(c_1\, r_1^{n+1}+c_2\, r_2^{n+1})=-c_1\, r_1^{n+2}-c_2 r_1\, r_2^{n+1}-c_1 r_2\, r_1^{n+1}-c_2\, r_2^{n+2}\\
\beta\, a_n&=r_1r_2(c_1\, r_1^n+c_2\, r_2^n)=c_1 r_2\, r_1^{n+1}+c_2 r_1\, r_2^{n+1}.
\end{aligned}$$

Direct summation implies

$$a_{n+2}+\alpha\, a_{n+1}+\beta\, a_n=0, \qquad n\geq 0.$$

Consequently, the solution of the differential equation is

$$y(x)=\sum_{n=0}^{\infty}(C_1\, r_1^n+c_2\, r_2^n)\, x^n$$

$$
\begin{aligned}
&= C_1 \sum_{n=0}^{\infty}\left(\frac{-\alpha-\sqrt{\alpha^2-4\beta}}{2}\right)^n x^n + C_2 \sum_{n=0}^{\infty}\left(\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2}\right)^n x^n \\
&= \frac{C_1}{1+\dfrac{\alpha+\sqrt{\alpha^2-4\beta}}{2}\, x} + \frac{C_2}{1+\dfrac{\alpha-\sqrt{\alpha^2-4\beta}}{2}\, x} \\
&= \frac{C_1}{1-r_1\, x} + \frac{C_2}{1-r_2\, x}
\end{aligned}
$$

It is not difficult to show that the set

$$\left\{\frac{1}{1-r_1\, x}, \frac{1}{1-r_2\, x}\right\}$$

indeed form a fundamental set of solutions, which can be shown by means of the definition. The set of solutions $\{y_1, y_2\}$ forms a fundamental set if the Wronskian

$$W(y_1,y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ \dfrac{dy_1}{dx} & \dfrac{dy_2}{dx} \end{vmatrix} = y_1(x)\frac{dy_2}{dx} - y_2(x)\frac{dy_1}{dx} \neq 0$$

For the mentioned solutions we note that

$$
\begin{aligned}
W\left(\frac{1}{1-r_1x}, \frac{1}{1-r_2x}\right) &= \begin{vmatrix} \frac{1}{1-r_1x} & \frac{1}{1-r_2x} \\ \frac{r_1}{(r_1x-1)^2} & \frac{r_2}{(r_2x-1)^2} \end{vmatrix} \\
&= \frac{r_2}{(1-r_1x)(1-r_2x)^2} - \frac{r_1}{(1-r_2x)(1-r_1x)^2} \\
&= \frac{r_2(1-r_1x)-r_1(1-r_1x)}{(1-r_1x)^2(1-r_2x)^2} \\
&= \frac{r_2-r_1}{(1-r_1x)^2(1-r_2x)^2} \neq 0.
\end{aligned}
$$

Then, (y_1, y_2) are linearly independent solutions. □

Note 1: The exact solutions of the differential equation

$$(1+\alpha\, x+\beta\, x^2)\, y''+(2\,\alpha+4\,\beta x)\, y'+2\,\beta\, y=0$$

can be written in terms of the Hypergeometric function as

$$y(x) = \,{}_2F_1\left(1,2;2;\frac{\beta x}{\sqrt{\alpha^2-4\beta}}+\frac{\alpha}{2\sqrt{\alpha^2-4\beta}}+\frac{1}{2}\right).$$

Note 2: The exact solutions of the differential equation

$$(1+\alpha\, x+\beta\, x^2)\, y''+\left(\frac{\alpha}{4}+\frac{\beta}{2}x\right)\, y'+\frac{\beta}{16}\, y=0$$

can be written in terms of the Hypergeometric function as

$$y(x) = \,{}_2F_1\left(-\frac{1}{4},-\frac{1}{4};\frac{1}{4};\frac{\beta x}{\sqrt{\alpha^2-4\beta}}+\frac{\alpha}{2\sqrt{\alpha^2-4\beta}}+\frac{1}{2}\right), \qquad \alpha>2\sqrt{\beta}.$$

5.1.2 subclass II: $(1-\,x^2)\,y''+(\gamma+\delta x)\,y'+\varepsilon\,y=0$

Lemma 5.1.5. *The power series $\sum_{n=0}^{\infty} a_n x^n$ is a solution of*

$$(1-\,x^2)\,y''+(\gamma+\delta x)\,y'+\varepsilon\,y=0 \tag{5.1.14}$$

if and only if the coefficients of the power series solution satisfy three-term recurrence relation

$$a_{n+2}+\frac{\gamma}{n+2}\,a_{n+1}+\frac{\varepsilon-\,n\,(n-1)+\delta\,n}{(n+2)(n+1)}\,a_n=0,\qquad n=0,1,\ldots, \tag{5.1.15}$$

with exact solutions given by

$$\begin{aligned}y(x)&=C_1{}_2F_1\left(-\frac{1+\delta+\sqrt{(1+\delta)^2+4\varepsilon}}{2},-\frac{1+\delta-\sqrt{(1+\delta)^2+4\varepsilon}}{2};-\frac{\gamma+\delta}{2};\frac{1-x}{2}\right)\\&+C_2\,x^{1+\dfrac{\gamma+\delta}{2}}\,{}_2F_1\left(\frac{\gamma+1+\sqrt{(1+\delta)^2+4\varepsilon}}{2},\frac{\gamma+1-\sqrt{(1+\delta)^2+4\varepsilon}}{2};2+\frac{\gamma+\delta}{2};\frac{1-x}{2}\right).\end{aligned} \tag{5.1.16}$$

5.2 Polynomial solutions

Theorem 5.2.1. *If*

$$\varepsilon=-\beta\,n\,(n-1)-\delta\,n, \tag{5.2.1}$$

the coefficients of the polynomial solution $y_n(x)=\sum_{k=0}^{n} a_k x^k$ of the differential equation

$$(1+\alpha\,x+\beta\,x^2)\,y''(x)+(\gamma+\delta\,x)\,y'(x)-n\,\left(\beta\,(n-1)+\delta\right)y(x)=0 \tag{5.2.2}$$

are given explicitly using the three-term recurrence relation

$$a_{k+2}+\frac{\gamma+\alpha\,k}{k+2}\,a_{k+1}+\frac{(k-n)(\delta+\beta(k+n-1))}{(k+2)(k+1)}\,a_k=0,\qquad k=0,1,\ldots,n \tag{5.2.3}$$

Setting $a_0=1$:

- Zero-degree polynomial solution: $n=0,\ k=0,$

$$y_0(x)=1. \tag{5.2.4}$$

- First-degree polynomial solution: $n=1,\ k=0,1,$ then $\dfrac{\gamma}{2}\,a_1-\dfrac{\delta}{2}\,a_0=0\Rightarrow a_1=\dfrac{\delta}{\gamma},$

$$y_1(x)=1+\frac{\delta}{\gamma}\,x. \tag{5.2.5}$$

- Second-degree polynomial solution: $n=2,\ k=0,1,2,$ then for $k=0,$

$$a_2+\frac{\gamma}{2}\,a_1-(\delta+\beta)\,a_0=0,\quad \frac{\gamma+\alpha}{3}a_2-\frac{(\delta+2\beta)}{6}a_1=0.$$

  Thus

$$y_2(x)=1+\frac{2(\beta+\delta)(\alpha+\gamma)}{2\beta+\delta+\gamma(\alpha+\gamma)}\,x+\frac{(\beta+\delta)(2\beta+\delta)}{2\beta+\delta+\gamma(\alpha+\gamma)}\,x^2. \tag{5.2.6}$$

- Third-degree polynomial solution: $n=3$, $k=0,1,2,3$, then for $k=0$,

$$-\frac{3}{2}(2\beta+\delta)a_0+\frac{1}{2}\gamma a_1+a_2=0,\quad -\frac{1}{3}(3\beta+\delta)a_1+\frac{1}{3}(\alpha+\gamma)a_2+a_3=0,$$
$$-\frac{1}{12}(4\beta+\delta)a_2+\frac{1}{4}(2\alpha+\gamma)a_3=0.$$

Thus

$$\begin{aligned}y_3(x)=1&+\frac{3(2\beta+\delta)(4\beta+\delta+(\alpha+\gamma)(2\alpha+\gamma))}{4\alpha(3\beta+\delta)+(2\alpha^2+10\beta+3\delta)\,\gamma+3\alpha\gamma^2+\gamma^3}\,x\\&+\frac{3(2\beta+\delta)(3\beta+\delta)(2\alpha+\gamma)}{4\alpha(3\beta+\delta)+(2\alpha^2+10\beta+3\delta)\,\gamma+3\alpha\gamma^2+\gamma^3}\,x^2\\&+\frac{(2\beta+\delta)(3\beta+\delta)(4\beta+\delta)}{4\alpha(3\beta+\delta)+(2\alpha^2+10\beta+3\delta)\,\gamma+3\alpha\gamma^2+\gamma^3}\,x^3.\end{aligned}\tag{5.2.7}$$

These polynomials can be generalized in terms of the Gauss Hypergeometric function

$$\begin{aligned}y_n(x)&=\left(-\sqrt{\alpha^2-4\beta}\right)^n\left(\frac{\alpha\,\delta-2\beta\,\gamma+\delta\,\sqrt{\alpha^2-4\beta}}{2\beta\sqrt{\alpha^2-4\,\beta}}\right)_n\\&\times\,{}_2F_1\left(-n,\frac{\delta}{\beta}+n-1;\frac{\alpha\,\delta-2\beta\,\gamma+\delta\,\sqrt{\alpha^2-4\beta}}{2\,\beta\,\sqrt{\alpha^2-4\,\beta}};\frac{\alpha+2\beta\,x}{2\sqrt{\alpha^2-4\,\beta}}+\frac{1}{2}\right),\quad n=0,1,\ldots.\end{aligned}\tag{5.2.8}$$

5.2.1 Subclass I: Extended Legendre differential equation

Theorem 5.2.2. *The coefficients of the polynomial solution $y_n(x)=\sum_{k=0}^n a_kx^k$ of the differential equation*

$$(1-\;x^2)\,y''+(\gamma+\delta x)\,y'-n(\delta-(n-1))\,y=0,\quad -1<x<1,\tag{5.2.9}$$

are given explicitly using the three-term recurrence relation

$$a_{k+2}+\frac{\gamma}{k+2}\,a_{k+1}+\frac{\delta\,(k-n)}{(k+2)(k+1)}\,a_k=0,\qquad k=0,1,\ldots,n\tag{5.2.10}$$

with polynomial solutions given explicitly by

$$y_n(x)=(-2)^n\left(-\frac{\gamma+\delta}{2}\right)_n\,{}_2F_1\left(-n,n-\delta-1;-\frac{\gamma+\delta}{2};\frac{1-x}{2}\right),\quad n=0,1,\ldots.\tag{5.2.11}$$

up to a constant.

5.2.2 Subclass II: $(1+\alpha\,x)\,y''(x)+(\gamma+\delta\,x)\,y'(x)-n\,\delta\,y(x)=0.$

Theorem 5.2.3. *The coefficients of the polynomial solution $y_n(x)=\sum_{k=0}^n a_kx^k$ of the differential equation*

$$(1+\alpha\,x)\,y''(x)+(\gamma+\delta\,x)\,y'(x)-n\,\delta\,y(x)=0\tag{5.2.12}$$

are given explicitly using the three-term recurrence relation

$$a_{k+2}+\frac{\gamma+\alpha\,k}{k+2}\,a_{k+1}+\frac{\delta\,(k-n)}{(k+2)(k+1)}\,a_k=0,\qquad k=0,1,\ldots,n\tag{5.2.13}$$

with polynomial solutions given explicitly by

$$y_n(x)=\alpha^n\left(\frac{\alpha\,\gamma-\delta}{\alpha^2}\right)_n {}_1F_1\left(-n;\frac{\alpha\,\gamma-\delta}{\alpha^2};-\frac{\delta}{\alpha}x-\frac{\delta}{\alpha^2}\right),\quad n=0,1,2,\ldots,\tag{5.2.14}$$

up to a constant.

5.2.3 Subclass III: Extended Hermite differential equation

Theorem 5.2.4. *The coefficients of the polynomial solution $y_n(x)=\sum_{k=0}^n a_k x^k$ of the differential equation*

$$y''(x)+(\gamma+\delta\,x)\,y'(x)-n\,\delta\,y(x)=0\tag{5.2.15}$$

are given explicitly using the three-term recurrence relation

$$a_{k+2}+\frac{\gamma}{k+2}\,a_{k+1}+\frac{\delta\,(k-n)}{(k+2)(k+1)}\,a_k=0,\qquad k=0,1,\ldots,n\tag{5.2.16}$$

with polynomial solutions given explicitly by

$$y_n(x)=(\delta x+\gamma)^n{}_2F_0\left(-\frac{n}{2},-\frac{n-1}{2};\;-\;;\frac{2\delta}{(\delta x+\gamma)^2}\right),\qquad n=0,1,2,\ldots,\tag{5.2.17}$$

up to a constant.

5.2.4 Subclass IV: $(1+\alpha\,x+\beta\,x^2)\,y''(x)+\gamma\,y'(x)-n\,(n-1)\,\beta\,y(x)=0.$

Theorem 5.2.5. *The coefficients of the polynomial solution $y_n(x)=\sum_{k=0}^n a_k x^k$ of the differential equation*

$$(1+\alpha x+\beta x^2)\,y''(x)+\gamma\,y'(x)-n(n-1)\,\beta\,y(x)=0\tag{5.2.18}$$

are given explicitly using the three-term recurrence relation

$$a_{k+2}+\frac{\gamma+\alpha\,k}{k+2}\,a_{k+1}+\frac{(k-n)(\beta(k+n-1))}{(k+2)(k+1)}\,a_k=0,\qquad k=0,1,\ldots,n\tag{5.2.19}$$

with polynomial solutions given explicitly by

$$y_n(x)={}_2F_1\left(-n,n-1;-\frac{\gamma}{\sqrt{\alpha^2-4\beta}};\frac{\alpha+2\beta x}{2\sqrt{\alpha^2-4\beta}}+\frac{1}{2}\right),\qquad n=0,1,2,\ldots,\tag{5.2.20}$$

up to a constant.

Chapter 6

On the solutions of the differential equation: $x\,(\alpha_1 + \alpha_2\,x)\,y'' + (\beta_0 + \beta_1\,x)\,y' + \varepsilon\,y = 0$

6.1 Series solutions

The point $x = 0$ is a singular point of the differential equation

$$x(\alpha_1 + \alpha_2\,x)\,y'' + (\beta_0 + \beta_1\,x)\,y' + \varepsilon\,y = 0 \tag{6.1.1}$$

where α_j, β_j, $(j = 0, 1, 2)$ and ε are real constants. If

$$x\left(\frac{\beta_0 + \beta_1\,x}{\alpha_1\,x + \alpha_2\,x^2}\right), \quad \textit{and} \quad x^2\left(\frac{\varepsilon}{\alpha_1\,x + \alpha_2\,x^2}\right)$$

are both analytic at the singular point $x = 0$, then the point $x = 0$ is called a regular singular point of the equation (6.1.1). For the regular singular point x_0, we have

$$\lim_{x\to 0} x\left(\frac{\beta_0 + \beta_1\,x}{\alpha_1\,x + \alpha_2\,x^2}\right) = \frac{\beta_0}{\alpha_1}, \tag{6.1.2}$$

and

$$\lim_{x\to 0} x^2\left(\frac{\varepsilon}{\alpha_1\,x + \alpha_2\,x^2}\right) = 0 \tag{6.1.3}$$

then the quadratic

$$r(r-1) + \frac{\beta_0}{\alpha_1} r = 0$$

is the indicial equation with roots $r = 0, r = 1 - \beta_0/\alpha_1$ are the indicial roots or the exponents of the regular singular point $x = 0$.

Theorem 6.1.1. *Let $r = 0$ be an exponent of the regular singular point $x = 0$. In the neighbourhood of the point $x = 0$, the coefficients $\{C_n\}_{n=0}^{\infty}$ of the series solution*

$$y(x) = \sum_{n=0}^{\infty} C_k\,x^k$$

of the differential equation

$$x\,(\alpha_1 + \alpha_2\,x)\,y'' + (\beta_0 + \beta_1\,x)\,y' + \varepsilon\,y = 0, \qquad \alpha_1 \neq 0 \tag{6.1.4}$$

satisfy a two-term recurrence relation

$$C_{k+1}=-\frac{[\alpha_2 k(k-1)+\beta_1 k+\varepsilon]}{[\alpha_1 k(k+1)+(k+1)\beta_0]}C_k,\quad k=0,1,2,\dots \tag{6.1.5}$$

where the constant C_0 is chosen arbitrarily.

Proof. In the neighbourhood of the regular singular point $x=0$, the series solution takes the form

$$y(x)=\sum_{k=0}^{\infty}C_k\,x^k,\quad y'(x)=\sum_{k=1}^{\infty}k\,C_k\,x^{k-1},\quad y''(x)=\sum_{k=2}^{\infty}k\,(k-1)\,C_k\,x^{k-2}.$$

The substitution of $y(x), y'(x)$ and $y''(x)$ in the given differential equation gives

$$(\alpha_1 x+\alpha_2 x^2)\sum_{k=2}^{\infty}k\,(k-1)\,C_k\,x^{k-2}+\ (\beta_0+\beta_1 x)\sum_{k=1}^{\infty}k\,C_k\,x^{k-1}+\varepsilon\sum_{k=0}^{\infty}C_k\,x^k=0.$$

The expansion of the infinite series implies

$$\begin{aligned}\alpha_1\sum_{k=2}^{\infty}k\,(k-1)\,C_k\,x^{k-1}&+\alpha_2\sum_{k=2}^{\infty}k\,(k-1)\,C_k\,x^k\\&+\beta_0\sum_{k=1}^{\infty}k\,C_k\,x^{k-1}+\beta_1\sum_{k=1}^{\infty}k\,C_k\,x^k+\varepsilon\sum_{k=0}^{\infty}C_k\,x^k=0.\end{aligned}$$

Shifting the indices then yields

$$\begin{aligned}\alpha_1\sum_{k=1}^{\infty}k\,(k+1)\,C_{k+1}\,x^k&+\alpha_2\sum_{k=2}^{\infty}k\,(k-1)\,C_k\,x^k\\&+\beta_0\sum_{k=0}^{\infty}(k+1)\,C_{k+1}\,x^k+\beta_1\sum_{k=1}^{\infty}k\,C_k\,x^k+\varepsilon\sum_{k=0}^{\infty}C_k\,x^k=0.\end{aligned}$$

To unify the indices, we write

$$\begin{aligned}2\alpha_1 C_2 x&+\alpha_1\sum_{k=2}^{\infty}k\,(k+1)\,C_{k+1}\,x^k+\alpha_2\sum_{k=2}^{\infty}k\,(k-1)\,C_k\,x^k\\&+\beta_0 C_1+2\beta_0 C_2 x+\beta_0\sum_{k=2}^{\infty}(k+1)\,C_{k+1}\,x^k+\beta_1 C_1 x+\beta_1\sum_{k=2}^{\infty}k\,C_k\,x^k\\&+\varepsilon C_0+\varepsilon C_1\,x+\varepsilon\sum_{k=2}^{\infty}C_k\,x^k=0.\end{aligned}$$

Comparing the coefficients of x:

$$\beta_0 C_1+\varepsilon C_0=0,\quad 2\alpha_1 C_2+(\beta_1+\varepsilon)C_1=0$$

and in general

$$[\alpha_1 k(k+1)+\beta_0(k+1)]C_{k+1}+[\alpha_2 k(k-1)+\beta_1 k+\varepsilon]C_k=0,\quad k=0,1,2,\dots$$

where C_0 is arbitrary. □

We may note recursively,

$$
\begin{aligned}
C_k &= -\frac{[\alpha_2(k-1)(k-2)+\beta_1(k-1)+\varepsilon]}{k[\alpha_1(k-1)+\beta_0]}C_{k-1}\\
&= (-1)^2\frac{[\alpha_2(k-1)(k-2)+\beta_1(k-1)+\varepsilon]}{k[\alpha_1(k-1)+\beta_0]}\frac{[\alpha_2(k-2)(k-3)+\beta_1(k-2)+\varepsilon]}{(k-1)[\alpha_1(k-2)+\beta_0]}C_{k-2}\\
&= (-1)^k\frac{[\alpha_2(k-1)(k-2)+\beta_1(k-1)+\varepsilon]}{k[\alpha_1(k-1)+\beta_0]}\times\frac{[\alpha_2(k-2)(k-3)+\beta_1(k-2)+\varepsilon]}{(k-1)[\alpha_1(k-2)+\beta_0]}\\
&\times\frac{[\alpha_2(k-3)(k-4)+\beta_1(k-3)+\varepsilon]}{(k-2)[\alpha_1(k-3)+\beta_0]}\cdots\times\frac{\varepsilon}{\beta_0}C_0
\end{aligned}
$$

that can be written using the Pochhammer symbol as

$$
C_k=\left(-\frac{\alpha_2}{\alpha_1}\right)^k\frac{\left(\dfrac{\beta_1-\alpha_2-\sqrt{\alpha_2^2-2(\beta_1+2\varepsilon)\alpha_2+\beta_1^2}}{2\alpha_2}\right)_k\left(\dfrac{\beta_1-\alpha_2+\sqrt{\alpha_2^2-2(\beta_1+2\varepsilon)\alpha_2+\beta_1^2}}{2\alpha_2}\right)_k}{\left(\dfrac{\beta_0}{\alpha_1}\right)_k k!}
$$

Theorem 6.1.2. *Let $r=0$ be an exponent of the regular singular point $x=0$. In the neighbourhood of the point $x=0$, the series solution of the differential equation*

$$x\,(\alpha_1+\alpha_2\,x)\,y''+(\beta_0+\beta_1\,x)\,y'+\varepsilon\,y=0$$

is given by

$$
\begin{aligned}
&y(x)\\
&={}_2F_1\left(\frac{\beta_1-\alpha_2-\sqrt{(\alpha_2-\beta_1)^2-4\varepsilon\alpha_2}}{2\alpha_2},\frac{\beta_1-\alpha_2+\sqrt{(\alpha_2-\beta_1)^2-4\varepsilon\alpha_2}}{2\alpha_2};\frac{\beta_0}{\alpha_1};-\frac{\alpha_2}{\alpha_1}x\right)
\end{aligned}
\tag{6.1.6}
$$

where the constant C_0 is arbitrary.

6.1.1 Subclass I: $\alpha_1\,x\,y''+(\beta_0+\beta_1\,x)\,y'+\varepsilon\,y=0$

For the differential equation

$$x\,\alpha_1\,y''+(\beta_0+\beta_1\,x)\,y'+\varepsilon\,y=0$$

the coefficients of the series solution assume the form

$$
\begin{aligned}
C_k &= -\frac{[\beta_1(k-1)+\varepsilon]}{k[\alpha_1(k-1)+\beta_0]}C_{k-1}\\
&= (-1)^2\frac{[\beta_1(k-1)+\varepsilon]}{k[\alpha_1(k-1)+\beta_0]}\frac{[\beta_1(k-2)+\varepsilon]}{(k-1)[\alpha_1(k-2)+\beta_0]}C_{k-2}\\
&= (-1)^k\frac{[\beta_1(k-1)+\varepsilon]}{k[\alpha_1(k-1)+\beta_0]}\times\frac{[\beta_1(k-2)+\varepsilon]}{(k-1)[\alpha_1(k-2)+\beta_0]}\frac{[\beta_1(k-3)+\varepsilon]}{(k-2)[\alpha_1(k-3)+\beta_0]}\cdots\times\frac{\varepsilon}{\beta_0}C_0
\end{aligned}
\tag{6.1.7}
$$

From which it follows that

$$
C_k=\frac{(-1)^k\left(\dfrac{\beta_1}{\alpha_1}\right)^k\left(\dfrac{\varepsilon}{\beta_1}\right)_k}{k!\left(\dfrac{\beta_0}{\alpha_1}\right)_k}.
\tag{6.1.8}
$$

Theorem 6.1.3. *The series solution of the differential equation*

$$x\,\alpha_1\,y''+(\beta_0+\beta_1\,x)\,y'+\varepsilon\,y=0$$

is given by

$$y(x)={}_1F_1\left(\frac{\varepsilon}{\beta_1};\frac{\beta_0}{\alpha_1};-\frac{\beta_1}{\alpha_1}x\right) \tag{6.1.9}$$

6.2 Polynomial solutions

Theorem 6.2.1. *The polynomial solutions of the differential equation*

$$x\,(\alpha_1+\alpha_2\,x)\,y''+(\beta_0+\beta_1\,x)\,y'-n\,(\alpha_2\,(n-1)+\beta_1)\,y=0, \tag{6.2.1}$$

are given by

$$y(x)={}_2F_1\left(-n,n-1+\frac{\beta_1}{\alpha_2};\frac{\beta_0}{\alpha_1};-\frac{\alpha_2}{\alpha_1}x\right). \tag{6.2.2}$$

Proof: The polynomial solutions occurs if

$$\frac{\beta_1-\alpha_2-\sqrt{(\alpha_2-\beta_1)^2-4\varepsilon\alpha_2}}{2\alpha_2}=-n,\qquad n=0,1,2,\ldots$$

Solving for ε yields

$$\varepsilon=-\,n\,(n-1)\,\alpha_2-n\,\beta_1.$$

6.3 Applications

6.3.1 Laguerre differential equation

Lemma 6.3.1.

$$\lim_{\epsilon\to0}\left(\frac{a}{\epsilon}\right)_k\left(\frac{\epsilon}{b}\right)^k=\left(\frac{a}{b}\right)^k. \tag{6.3.1}$$

Proof: Using the definition of Pochhammer symbol:

$$\begin{aligned}\lim_{\epsilon\to0}\left(\frac{a}{\epsilon}\right)_k\left(\frac{\epsilon}{b}\right)^k&=\lim_{\epsilon\to0}\left(\frac{a}{\epsilon}\right)\left(\frac{a}{\epsilon}+1\right)\left(\frac{a}{\epsilon}+2\right)\ldots\left(\frac{a}{\epsilon}+k-1\right)\left(\frac{\epsilon}{b}\right)^k\\&=\lim_{\epsilon\to0}a\,(a+\epsilon)\,(a+2\epsilon)\ldots(a+(k-1)\epsilon)\left(\frac{1}{b}\right)^k=\left(\frac{a}{b}\right)^k.\end{aligned}$$

Theorem 6.3.2. *The polynomial solutions of the differential equation*

$$\alpha_1 x\,y''+(\beta_0+\beta_1\,x)\,y'-n\,\beta_1\,y=0 \tag{6.3.2}$$

are given, in terms of the confluent hypergeometric functions, by

$$y(x)={}_1F_1\left(-n;\frac{\beta_0}{\alpha_1};-\frac{\beta_1}{\alpha_1}x\right),\qquad n=0,1,2,\ldots, \tag{6.3.3}$$

up to a constant.

6.3.2 Bessel differential equation

Theorem 6.3.3. *The polynomial solutions of the differential equation*

$$\alpha_2\, x^2\, y'' + (\beta_0 + \beta_1\, x)\, y' - (n(n-1)\alpha_2 + n\,\beta_1)\, y = 0 \tag{6.3.4}$$

are given by

$$y(x) = {}_2F_0\left(-n, n - 1 + \frac{\beta_1}{\alpha_2}; -; -\frac{\alpha_2}{\beta_0} x\right), \qquad n = 0, 1, 2, \dots, \tag{6.3.5}$$

up to a constant.

Chapter 7

On the solutions of the differential equation: $(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)\, y'' + (\beta_0 + \beta_1 x + \beta_2 x^2)\, y' + (\varepsilon_0 + \varepsilon_1 x)\, y = 0$

7.1 Series solutions

Theorem 7.1.1. *In the neighbourhood of the ordinary point $x = 0$, the coefficients $\{C_k\}_{k=0}^{\infty}$ of the series solution $y(x) = \sum_{k=0}^{\infty} C_k x^k$ of the differential equation*

$$(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)\, y'' + (\beta_0 + \beta_1 x + \beta_2 x^2)\, y' + (\varepsilon_0 + \varepsilon_1 x)\, y = 0, \tag{7.1.1}$$

satisfy the four-term recurrence relation

$$\begin{aligned} \alpha_0 (k+3)\,(k+2)\, C_{k+3} + (k+2)(\alpha_1\,(k+1) + \beta_0)\, C_{k+2} + (\alpha_2 k\,(k+1) + \beta_1\,(k+1) + \varepsilon_0)\, C_{k+1} \\ + (\alpha_3 k(k-1) + \beta_2\, k + \varepsilon_1) C_k = 0, \quad k = 0, 1, 2, \ldots . \end{aligned} \tag{7.1.2}$$

where, by assumption, $C_{-1} = 0$.

Proof: Assume a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} C_k\, x^k, \quad y'(x) = \sum_{n=1}^{\infty} k C_k\, x^{k-1}, \quad y''(x) = \sum_{n=0}^{\infty} k(k-1) C_k\, x^{k-2},$$

using these expressions of y, y' and y'', equation (7.1.1) yields

$$\begin{aligned} (\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) \sum_{n=2}^{\infty} k(k-1) C_k\, x^{k-2} + (\beta_0 + \beta_1 x + \beta_2 x^2) \sum_{n=1}^{\infty} k C_k\, x^{k-1} \\ + (\varepsilon_0 + \varepsilon_1 x) \sum_{n=0}^{\infty} C_k\, x^k = 0. \end{aligned}$$

From which, it follows by combining sums,

$$\alpha_0 \sum_{n=2}^{\infty} k(k-1) C_k\, x^{k-2} + \alpha_1 \sum_{n=2}^{\infty} k(k-1) C_k\, x^{k-1} + \alpha_2 \sum_{n=2}^{\infty} k(k-1) C_k\, x^k + \alpha_3 \sum_{n=2}^{\infty} k(k-1) C_k\, x^{k+1}$$

$$+\beta_0\sum_{n=1}^{\infty}kC_k\,x^{k-1}+\beta_1\sum_{n=1}^{\infty}kC_k\,x^{k}+\beta_2\sum_{n=1}^{\infty}kC_k\,x^{k+1}+\varepsilon_0\sum_{n=0}^{\infty}C_k\,x^{k}+\varepsilon_1\sum_{n=0}^{\infty}C_k\,x^{k+1}=0.$$

Adjusting the indices, we obtain

$$\begin{aligned}
&2\alpha_0\,C_2+6\alpha_0\,C_3\,x+12\alpha_0\,C_4\,x^2\\
&\quad+\alpha_0\sum_{n=3}^{\infty}(k+2)(k+1)\,C_{k+2}\,x^k+2\alpha_1\,C_2x+6\alpha_1C_3\,x^2+\alpha_1\sum_{n=3}^{\infty}k(k+1)\,C_{k+1}\,x^k\\
&\quad+2\,\alpha_2\,C_2\,x^2+\alpha_2\sum_{n=2}^{\infty}k(k-1)\,C_k\,x^k+\alpha_3\sum_{n=3}^{\infty}(k-1)(k-2)C_{k-1}\,x^k+\beta_0\,C_1+2\beta_0\,C_2x\\
&\quad+3\,\beta_0\,C_3\,x^2+\beta_0\sum_{n=3}^{\infty}(k+1)\,C_{k+1}\,x^k+\beta_1\,C_1x+2\beta_1\,C_2\,x^2+\beta_1\sum_{n=3}^{\infty}k\,C_k\,x^k\\
&\quad+\beta_2\,C_1\,x^2+\beta_2\sum_{n=3}^{\infty}(k-1)C_{k-1}\,x^k+\varepsilon_0\,C_0+\varepsilon_0\,C_1x+\varepsilon_0\,C_2\,x^2+\varepsilon_0\sum_{n=3}^{\infty}C_k\,x^k\\
&\quad+\varepsilon_1\,C_0+\varepsilon_1\,C_1+\varepsilon_1\sum_{n=3}^{\infty}C_{k-1}\,x^k=0.
\end{aligned}$$

This implies

$$\begin{aligned}
2\alpha_0C_2+\beta_0C_1+\varepsilon_0C_0&=0,\\
6\alpha_0C_3+2(\alpha_1+\beta_0)\,C_2+(\beta_1+\varepsilon_0)C_1+\varepsilon_1C_0&=0,\\
12\alpha_0C_4+(6\alpha_1+3\beta_0)C_3+(2\alpha_2+2\beta_1+\varepsilon_0)C_2+(\beta_2+\varepsilon_1)C_1&=0,
\end{aligned}$$

and for arbitrary k,

$$\begin{aligned}
\alpha_0(k+2)(k+1)C_{k+2}+(k+1)(\alpha_1k+\beta_0)C_{k+1}+(\alpha_2k(k-1)+\beta_1\,k+\varepsilon_0)C_k&\\
+(\alpha_3(k-1)(k-2)+\beta_2\,(k-1)+\varepsilon_1)C_{k-1}=0.&
\end{aligned}\tag{7.1.3}$$

The coefficients in the four-term recurrence relation of the series solution of the differential equation satisfy

$$\begin{aligned}
\alpha_0(k+3)\,(k+2)\,C_{k+3}+(k+2)(\alpha_1\,(k+1)+\beta_0)\,C_{k+2}+(\alpha_2k\,(k+1)+\beta_1\,(k+1)+\varepsilon_0)\,C_{k+1}&\\
+(\alpha_3k(k-1)+\beta_2\,k+\varepsilon_1)C_k=0,&
\end{aligned}$$

which gives

$$\begin{aligned}
C_{k+3}=-\frac{(\alpha_1\,(k+1)+\beta_0)}{\alpha_0(k+3)}\,C_{k+2}&-\frac{(\alpha_2k\,(k+1)+\beta_1\,(k+1)+\varepsilon_0)}{\alpha_0(k+3)\,(k+2)}C_{k+1}\\
&-\frac{(\alpha_3k(k-1)+\beta_2\,k+\varepsilon_1)}{\alpha_0(k+3)\,(k+2)}C_k.\quad k\neq-2,-3
\end{aligned}\tag{7.1.4}$$

For $k=-1$,

$$C_2=-\frac{\beta_0}{2\alpha_0}\,C_1-\frac{\varepsilon_0}{2\alpha_0}C_0,\tag{7.1.5}$$

while for $k=0$ we obtain

$$C_3=-\frac{(\alpha_1\;+\beta_0)}{3\,\alpha_0}\,C_2-\frac{(\beta_1+\varepsilon_0)}{6\,\alpha_0}C_1-\frac{\varepsilon_1}{6\,\alpha_0}C_0,\tag{7.1.6}$$

7.1.1 Subclass I: $\alpha_0 = 0$

In the case the differential equation

$$x(\alpha_1+\alpha_2\,x+\alpha_3x^2)\,y''+(\beta_0+\beta_1\,x+\beta_2\,x^2)\,y'+(\varepsilon_0+\varepsilon_1\,x)\,y=0, \tag{7.1.7}$$

has a regular singular point at $x=0$ because

$$\lim_{x\to 0} x\times\frac{\beta_0+\beta_1\,x+\beta_2\,x^2}{\alpha_1\,x+\alpha_2\,x^2+\alpha_3x^3}=\frac{\beta_0}{\alpha_1},\qquad \lim_{x\to 0} x^2\times\frac{\varepsilon_0+\varepsilon_1\,x}{\alpha_1\,x+\alpha_2\,x^2+\alpha_3x^3}=0.$$

The indicial equation

$$r^2-\left(1-\frac{\beta_0}{\alpha_1}\right)\,r=0 \tag{7.1.8}$$

has one of the exponents $r=0$. Thus, the series solution, in the neighbourhood of $x=0$ takes the form $y(x)=\sum_{k=0}^{\infty} C_k x^k$, and similar to the Theorem 7.1.1, the coefficients $\{C_k\}_{k=0}^{\infty}$ satisfy the three term recurrence relation

$$\begin{aligned}(k+2)(\alpha_1\,(k+1)+\beta_0)\,C_{k+2}&+(\alpha_2k\,(k+1)+\beta_1\,(k+1)+\varepsilon_0)\,C_{k+1}\\ &+(\alpha_3k(k-1)+\beta_2\,k+\varepsilon_1)C_k=0,\quad k=0,1,2,\ldots.\end{aligned} \tag{7.1.9}$$

or, by shifting $k\to k-1$,

$$\begin{aligned}(k+1)(\alpha_1\,k+\beta_0)\,C_{k+1}&+(\alpha_2k\,(k-1)+\beta_1\,k+\varepsilon_0)\,C_k\\ &+(\alpha_3(k-1)(k-2)+\beta_2\,(k-1)+\varepsilon_1)C_{k-1}=0,\quad k=0,1,2,\ldots.\end{aligned} \tag{7.1.10}$$

where

$$\beta_0\,C_1+\varepsilon_0\,C_0=0\Longrightarrow C_1=-\frac{\varepsilon_0}{\beta_0}C_0=-\frac{\mathcal{P}_1(\varepsilon_0)}{\beta_0}C_0,\qquad \mathcal{P}_1(\varepsilon_0)=\varepsilon_0. \tag{7.1.11}$$

Recursively, it follows

$$C_2=(-1)^2\frac{\varepsilon_0(\beta_1+\varepsilon_0)-\beta_0\varepsilon_1}{2\beta_0(\alpha_1+\beta_0)}=(-1)^2\frac{(\beta_1+\varepsilon_0)\mathcal{P}_1(\varepsilon_0)-\beta_0\varepsilon_1\mathcal{P}_0(\varepsilon_0)}{2!\alpha_1^2\left(\dfrac{\beta_0}{\alpha_1}\right)_2}=(-1)^2\frac{\mathcal{P}_2(\varepsilon_0)}{2!\,\alpha_1^2\left(\dfrac{\beta_0}{\alpha_1}\right)_2}, \tag{7.1.12}$$

where

$$\begin{aligned}&\mathcal{P}_2(\varepsilon_0)=(\beta_1+\varepsilon_0)\mathcal{P}_1(\varepsilon_0)-\beta_0\varepsilon_1\mathcal{P}_0(\varepsilon_0),\\ &\mathcal{P}_1(\varepsilon_0)=\varepsilon_0,\qquad \mathcal{P}_0(\varepsilon_0)=1.\end{aligned} \tag{7.1.13}$$

Further,

$$C_3=(-1)^3\frac{\varepsilon_0(-2(\alpha_1+\beta_0)\beta_2+(\beta_1+\varepsilon_0)(2(\alpha_2+\beta_1)+\varepsilon_0))-(2\beta_0(\alpha_2+\beta_1)+2\alpha_1\varepsilon_0+3\beta_0\varepsilon_0)\varepsilon_1}{6\beta_0(\alpha_1+\beta_0)(2\alpha_1+\beta_0)} \tag{7.1.14}$$

This coefficient can be written as

$$C_3=(-1)^3\frac{\mathcal{P}_3(\varepsilon_0)}{3!\alpha_1^3\left(\dfrac{\beta_0}{\alpha_1}\right)_3}, \tag{7.1.15}$$

where

$$\begin{aligned}\mathcal{P}_3(\varepsilon_0)&=(2\alpha_2+2\beta_1+\varepsilon_0)\mathcal{P}_2(\varepsilon_0)-2(\alpha_1+\beta_0)(\beta_2+\varepsilon_1)\mathcal{P}_1(\varepsilon_0)\\ \mathcal{P}_2(\varepsilon_0)&=(\beta_1+\varepsilon_0)\mathcal{P}_1(\varepsilon_0)-\beta_0\varepsilon_1\mathcal{P}_0(\varepsilon_0),\quad \mathcal{P}_1(\varepsilon_0)=\varepsilon_0.\end{aligned}\tag{7.1.16}$$

Similarly,

$$C_4=(-1)^4\,\frac{\mathcal{P}_4(\varepsilon_0)}{4!\,\alpha_1^4\left(\dfrac{\beta_0}{\alpha_1}\right)_4},\tag{7.1.17}$$

where

$$\begin{aligned}\mathcal{P}_4(\varepsilon_0)&=(6\alpha_2+3\beta_1+\varepsilon_0)\mathcal{P}_3(\varepsilon_0)-3(2\alpha_1+\beta_0)(2(\alpha_3+\beta_2)+\varepsilon_1)\mathcal{P}_2(\varepsilon_0)\\ \mathcal{P}_3(\varepsilon_0)&=(2\alpha_2+2\beta_1+\varepsilon_0)\mathcal{P}_2(\varepsilon_0)-2(\alpha_1+\beta_0)(\beta_2+\varepsilon_1)\mathcal{P}_1(\varepsilon_0)\\ \mathcal{P}_2(\varepsilon_0)&=(\beta_1+\varepsilon_0)\mathcal{P}_1(\varepsilon_0)-\beta_0\varepsilon_1\mathcal{P}_0(\varepsilon_0).\end{aligned}\tag{7.1.18}$$

Furthermore,

$$C_5=(-1)^5\,\frac{\mathcal{P}_5(\varepsilon_0)}{5!\,\alpha_1^5\left(\dfrac{\beta_0}{\alpha_1}\right)_5},\tag{7.1.19}$$

where

$$\begin{aligned}\mathcal{P}_5(\varepsilon_0)&=(12\alpha_2+4\beta_1+\varepsilon_0)\mathcal{P}_4(\varepsilon_0)-4(3\alpha_1+\beta_0)(6\alpha_3+3\beta_2+\varepsilon_1)\mathcal{P}_3(\varepsilon_0)\\ \mathcal{P}_4(\varepsilon_0)&=(6\alpha_2+3\beta_1+\varepsilon_0)\mathcal{P}_3(\varepsilon_0)-3(2\alpha_1+\beta_0)(2\alpha_3+2\beta_2+\varepsilon_1)\mathcal{P}_2(\varepsilon_0)\\ \mathcal{P}_3(\varepsilon_0)&=(2\alpha_2+2\beta_1+\varepsilon_0)\mathcal{P}_2(\varepsilon_0)-2(\alpha_1+\beta_0)(\beta_2+\varepsilon_1)\mathcal{P}_1(\varepsilon_0).\end{aligned}\tag{7.1.20}$$

In general,

$$C_k=(-1)^k\,\frac{\mathcal{P}_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k},\tag{7.1.21}$$

where

$$\begin{aligned}\mathcal{P}_k(\varepsilon_0)&=((k-1)(k-2)\alpha_2+(k-1)\beta_1+\varepsilon_0)\mathcal{P}_{k-1}(\varepsilon_0)\\ &\quad-(k-1)((k-2)\alpha_1+\beta_0)((k-2)(k-3)\alpha_3+(k-2)\beta_2+\varepsilon_1)\mathcal{P}_{k-2}(\varepsilon_0).\end{aligned}\tag{7.1.22}$$

Theorem 7.1.2. *In the neighbourhood of the regular singular point $x=0$, the series solution $y(x)=\sum_{k=0}^{\infty}C_kx^k$ of the differential equation*

$$x\,(\alpha_1+\alpha_2\,x+\alpha_3x^2)\,y''+(\beta_0+\beta_1\,x+\beta_2\,x^2)\,y'+(\varepsilon_0+\varepsilon_1\,x)\,y=0,\tag{7.1.23}$$

is given (up to a constant) explicitly by

$$y(x)=\sum_{k=0}^{\infty}(-1)^k\,\frac{\mathcal{P}_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k},\tag{7.1.24}$$

where

$$\begin{aligned}\mathcal{P}_k(\varepsilon_0)&=((k-1)(k-2)\alpha_2+(k-1)\beta_1+\varepsilon_0)\mathcal{P}_{k-1}(\varepsilon_0)\\ &\quad-(k-1)((k-2)\alpha_1+\beta_0)((k-2)(k-3)\alpha_3+(k-2)\beta_2+\varepsilon_1)\mathcal{P}_{k-2}(\varepsilon_0),\end{aligned}\tag{7.1.25}$$

where $C_0=1$.

7.1.2 Subclass II: $\alpha_0=\alpha_2=0$

Theorem 7.1.3. *In the neighbourhood of the regular singular point $x=0$, the series solution $y(x)=\sum_{k=0}^{\infty}C_kx^k$ of the differential equation*

$$x\,(\alpha_1+\alpha_3x^2)\,y''+(\beta_0+\beta_1\,x+\beta_2\,x^2)\,y'+(\varepsilon_0+\varepsilon_1\,x)\,y=0, \tag{7.1.26}$$

is given (up to a constant) explicitly by

$$y(x)=\sum_{k=0}^{\infty}(-1)^k\,\frac{s_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k}x^k, \tag{7.1.27}$$

where

$$\begin{aligned} s_k(\varepsilon_0)&=((k-1)\beta_1+\varepsilon_0)\,s_{k-1}(\varepsilon_0)\\ &\quad-(k-1)((k-2)\alpha_1+\beta_0)((k-2)(k-3)\alpha_3+(k-2)\beta_2+\varepsilon_1)\;\;s_{k-2}(\varepsilon_0), \end{aligned} \tag{7.1.28}$$

where $C_0=1$.

7.1.3 Subclass III: $\alpha_0=\alpha_3=0$

Theorem 7.1.4. *In the neighbourhood of the regular singular point $x=0$, the series solution $y(x)=\sum_{k=0}^{\infty}C_kx^k$ of the differential equation*

$$x\,(\alpha_1+\alpha_2\,x)\,y''+(\beta_0+\beta_1\,x+\beta_2\,x^2)\,y'+(\varepsilon_0+\varepsilon_1\,x)\,y=0, \tag{7.1.29}$$

is given (up to a constant) explicitly by

$$y(x)=\sum_{k=0}^{\infty}(-1)^k\,\frac{t_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k}x^k, \tag{7.1.30}$$

where

$$\begin{aligned} t_k(\varepsilon_0)&=((k-1)(k-2)\alpha_2+(k-1)\beta_1+\varepsilon_0)t_{k-1}(\varepsilon_0)\\ &\qquad-(k-1)((k-2)\alpha_1+\beta_0)((k-2)\beta_2+\varepsilon_1)t_{k-2}(\varepsilon_0), \end{aligned} \tag{7.1.31}$$

where $C_0=1$.

7.1.4 Subclass IV: $\alpha_0=\alpha_2=\alpha_3=0$

Theorem 7.1.5. *In the neighbourhood of the regular singular point $x=0$, the series solution $y(x)=\sum_{k=0}^{\infty}C_kx^k$ of the differential equation*

$$\alpha_1\,x\,y''+(\beta_0+\beta_1\,x+\beta_2\,x^2)\,y'+(\varepsilon_0+\varepsilon_1\,x)\,y=0, \tag{7.1.32}$$

is given (up to a constant) explicitly by

$$y(x)=\sum_{k=0}^{\infty}(-1)^k\,\frac{u_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k}x^k, \tag{7.1.33}$$

where

$$u_k(\varepsilon_0)=((k-1)\beta_1+\varepsilon_0)u_{k-1}(\varepsilon_0)-(k-1)((k-2)\alpha_1+\beta_0)((k-2)\beta_2+\varepsilon_1)u_{k-2}(\varepsilon_0), \tag{7.1.34}$$

where $C_0=1$.

7.1.5 Subclass V: $\alpha_0=\alpha_2=\alpha_3=\beta_1=0$

Theorem 7.1.6. *In the neighbourhood of the regular singular point $x=0$, the series solution $y(x)=\sum_{k=0}^{\infty} C_k x^k$ of the differential equation*

$$\alpha_1 x\,y''+(\beta_0+\beta_2\,x^2)\,y'+(\varepsilon_0+\varepsilon_1\,x)\,y=0, \tag{7.1.35}$$

is given (up to a constant) explicitly by

$$y(x)=\sum_{k=0}^{\infty}(-1)^k\,\frac{v_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k}x^k, \tag{7.1.36}$$

where

$$v_k(\varepsilon_0)=\varepsilon_0 v_{k-1}(\varepsilon_0)-(k-1)((k-2)\alpha_1+\beta_0)((k-2)\beta_2+\varepsilon_1)v_{k-2}(\varepsilon_0), \tag{7.1.37}$$

where $C_0=1$.

7.2 Polynomial solutions

Theorem 7.2.1. *The necessary and sufficient conditions for n-degree polynomial solutions to the second-order linear differential equation*

$$(\alpha_3\,x^3+\alpha_2\,x^2+\alpha_1\,x+\alpha_0)\;y''(x)+(\beta_2\,x^2+\beta_1\,x+\beta_0)\;y'(x)+(\varepsilon_1\,x+\varepsilon_0)\;y(x)=0, \tag{7.2.1}$$

(where $a_{k,j}, k=3,2,1, j=0,1,2,3$ are constants) that

$$\varepsilon_1=-n\,(n-1)\;\alpha_3-n\;\beta_2,\quad n=0,1,2,\ldots, \tag{7.2.2}$$

provided $\alpha_3^2+\beta_2^2\neq 0$ and the vanishing of $(n+1)\times(n+1)$-determinant Δ_{n+1} is given by

$$\Delta_{n+1}\;=\;\begin{vmatrix} S_0 & T_1 & \eta_1 & & & & \\ \gamma_1 & S_1 & T_2 & \eta_2 & & & \\ & \gamma_2 & S_2 & T_3 & \eta_3 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \gamma_{n-2} & S_{n-2} & T_{n-1} & \eta_{n-1} \\ & & & & \gamma_{n-1} & S_{n-1} & T_n \\ & & & & & \gamma_n & S_n \end{vmatrix}\;=\;0$$

where all the other entries are zeros and

$$\begin{aligned} S_k&=-\varepsilon_0-k\,((k-1)\,\alpha_2+\beta_1),\\ T_k&=-k\,((k-1)\,\alpha_1+\beta_0),\\ \gamma_k&=-\varepsilon_1-(k-1)\,((k-2)\,\alpha_3+\beta_2),\\ \eta_k&=-k\,(k+1)\,\alpha_0. \end{aligned} \tag{7.2.3}$$

Here ε_1 is fixed for a given n in the determinant $\Delta_{n+1}=0$ (the degree of the polynomial solution).

For example, for $n=0$,

$$\varepsilon_1 = 0 \quad \text{and} \quad \Delta_1 = \varepsilon_0 = 0, \tag{7.2.4}$$

for $n=1$,

$$\varepsilon_1 = -\beta_2 \quad \text{and} \quad \Delta_2 = \begin{vmatrix} -\varepsilon_0 & -\beta_0 \\ -\varepsilon_1 & -\varepsilon_0-\beta_1 \end{vmatrix} = 0, \tag{7.2.5}$$

for $n=2$,

$$\varepsilon_1 = -2\alpha_3-2\beta_2 \quad \text{and} \quad \Delta_3 = \begin{vmatrix} -\varepsilon_0 & -\beta_0 & -2\alpha_0 \\ -\varepsilon_1 & -\varepsilon_0-\beta_1 & -2(\alpha_1+\beta_0) \\ 0 & -\varepsilon_1-\beta_2 & -\varepsilon_0-2(\alpha_2+\beta_1) \end{vmatrix} = 0, \tag{7.2.6}$$

and for $n=3$,

$$\varepsilon_1 = -6\alpha_3-3\beta_2,$$
$$\Delta_4 = \begin{vmatrix} -\varepsilon_0 & -\beta_0 & -2\alpha_0 & 0 \\ -\varepsilon_1 & -\varepsilon_0-\beta_1 & -2(\alpha_1+\beta_0) & -6\alpha_0 \\ 0 & -\varepsilon_1-\beta_2 & -\varepsilon_0-2(\alpha_2+\beta_1) & -3(2\alpha_1+\beta_0) \\ 0 & 0 & -\varepsilon_1-2(\alpha_1+\beta_2) & -\varepsilon_0-3(2\alpha_2+\beta_1) \end{vmatrix} = 0. \tag{7.2.7}$$

Further, the coefficients of the polynomial solutions $y_n(x)=\sum_{k=0}^n C_k\,x^k$ satisfy the four-term recursive relation:

$$\begin{aligned} \alpha_0(k+3)\,(k+2)\,C_{k+3} &+ (k+2)(\alpha_1\,(k+1)+\beta_0)\,C_{k+2} + (k\,(k+1)\,\alpha_2+\beta_1\,(k+1)+\varepsilon_0)\,C_{k+1} \\ &+ ((k\,(k-1)-n\,(n-1))\;\alpha_3+(k-n)\;\beta_2)\,C_k = 0, \end{aligned} \tag{7.2.8}$$

where $k=0,1,2,\ldots,n$ and $C_{-1}=0$.

7.2.1 Subclass I: $\alpha_0 = 0$

In this case, the second-order linear differential equation

$$x(\alpha_3\,x^2+\alpha_2\,x+\alpha_1)\;y''(x)+(\beta_2 x^2+\beta_1 x+\beta_0)\;y'(x)-(\varepsilon_1 x+\varepsilon_0)\;y(x)=0, \tag{7.2.9}$$

has a polynomial solution of degree n if

$$\varepsilon_1 = n(n-1)\;\alpha_3+n\;\beta_2, \quad n=0,1,2,\ldots, \tag{7.2.10}$$

along with the vanishing of $(n+1)\times(n+1)$-determinant Δ_{n+1} given by

$$\Delta_{n+1} \;=\; \begin{vmatrix} S_0 & T_1 & & & & & \\ \gamma_1 & S_1 & T_2 & & & & \\ & \gamma_2 & S_2 & T_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma_{n-2} & S_{n-2} & T_{n-1} & \\ & & & & \gamma_{n-1} & S_{n-1} & T_n \\ & & & & & \gamma_n & S_n \end{vmatrix} \;=\; 0$$

where all the other entries are zeros and

$$\begin{aligned}
S_k &= -\varepsilon_0 - k\,((k-1)\,\alpha_2+\beta_1),\\
T_k &= -k\,((k-1)\,\alpha_1+\beta_0),\\
\gamma_k &= -\varepsilon_1-(k-1)\,((k-2)\,\alpha_3+\beta_2).
\end{aligned} \tag{7.2.11}$$

Here ε_1 is fixed for a given n in the determinant $\Delta_{n+1}=0$ (the degree of the polynomial solution).

Further the coefficients of the polynomial solutions

$$y_n(r)=\sum_{k=0}^{n} C_k x^k$$

satisfy the three-term recursive relation

$$\begin{aligned}
(k+2)(\alpha_1\,(k+1)+\beta_0)\,C_{k+2}+(k\,(k+1)\,\alpha_2+\beta_1\,(k+1)+\varepsilon_0)\,C_{k+1}&\\
+\,((k\,(k-1)-n\,(n-1))\ \alpha_3+(k-n)\ \beta_2)\,C_k=0,&
\end{aligned} \tag{7.2.12}$$

where $k=0,1,2,\ldots,n$ and $C_{-1}=0$. This recurrence relation can be written in a more compact form as

$$\gamma_k C_{k-1}+S_k c_k+T_{k+1}C_{k+1}=0,\quad C_0=1,\quad C_{-1}=0,\ \ n=0,1,2,\ldots. \tag{7.2.13}$$

where γ_n, S_n and T_n are given by (7.2.11). We can easily show that the determinant satisfies a three-term recurrence relation

$$\Delta_k=S_{k-1}\Delta_{k-1}-\gamma_{k-1}T_{k-1}\Delta_{k-2},\quad \Delta_0=1,\ \ \Delta_{-1}=0,\quad k=1,2,\ldots \tag{7.2.14}$$

which can be used to compute the determinant Δ_k recursively in terms of lower order determinants. In this case, however, we must fix n for each of the sub-determinants used in computing (7.2.11). For example, in the case of $n=1$ (corresponding to a polynomial solution of degree one), we have

$$\Delta_2\ =\begin{vmatrix} -\varepsilon_0 & -\beta_0\\ -\varepsilon_1 & -\varepsilon_0-\beta_1\end{vmatrix}=-\varepsilon_0(-\varepsilon_0-\beta_1)-\varepsilon_1\beta_0,$$

Meanwhile

$$S_1\Delta_1-\gamma_1T_1\Delta_0=(\varepsilon_0+\beta_0)\varepsilon_0+\varepsilon_1\beta_0,$$

as is expected. For $n=2$ (corresponding to a second-degree polynomial solution)

$$\Delta_3=\begin{vmatrix} -\varepsilon_0 & -\beta_0 & 0\\ -\varepsilon_1 & \varepsilon_0-\beta_1 & -2(\alpha_1+\beta_0)\\ 0 & -\varepsilon_1-\beta_2 & \varepsilon_0-2(\alpha_2+\beta_1)\end{vmatrix}=S_2\Delta_2-\gamma_2T_2\Delta_1,$$

if we expand the determinant along the last row. Note that if

$$T_{k+1}\gamma_{k+1}>0,\quad 0\le k\le n-1,$$

then all the $n+1$ roots of its determinant (7.2.11) are real and different.

7.2.2 Subclass II: $\alpha_0 = 0, \quad \alpha_3+\alpha_2+\alpha_1=0$

In this case, the differential equation reads,

$$x(\alpha_3 x^2+\alpha_2 x+\alpha_1)\;y''(x)+(\beta_2 x^2+\beta_1 x+\beta_0)\;y'(x)-(\varepsilon_1 x+\varepsilon_0)\;y(x)=0, \tag{7.2.15}$$

which can be written as

$$y''(x)+\left(\frac{\beta_0}{\alpha_1\;x}-\frac{\beta_2+\beta_1+\beta_0}{\alpha_3(\frac{\alpha_1}{\alpha_3}-1)\;(x-1)}+\frac{\frac{\beta_2\alpha_1^2}{\alpha_3^2}+\frac{\beta_1\alpha_1}{\alpha_3}+\beta_0}{\alpha_1(\frac{\alpha_1}{\alpha_3}-1)\;(x-\frac{\alpha_1}{\alpha_3})}\right)y'(x)$$
$$+\left(\frac{-(n(n-1)\alpha_3+n\beta_2)x-\varepsilon_0}{\alpha_3 x(x-1)(x-\frac{\alpha_1}{\alpha_3})}\right)y(x)=0 \tag{7.2.16}$$

or, simply

$$y''(x)+\left(\frac{\gamma}{x}+\frac{\delta}{(x-1)}+\frac{\epsilon}{(x-b)}\right)y'(x)+\left(\frac{\alpha\beta\;x-q}{x(x-1)(x-b)}\right)y(x)=0. \tag{7.2.17}$$

where

$$\begin{cases}\gamma=\dfrac{\beta_0}{\alpha_1}\\[2mm] \delta=\dfrac{\beta_2+\beta_1+\beta_0}{\alpha_3-\alpha_1}\\[2mm] \epsilon=\dfrac{\beta_2\alpha_1^2+\beta_1\alpha_1\alpha_3+\beta_0\alpha_3^2}{\alpha_3\alpha_1(\alpha_1-\alpha_3)}\\[2mm] \alpha=\dfrac{\beta_2+(n-1)\alpha_3}{\alpha_3}\\[2mm] \beta=-n\\[2mm] q=\dfrac{\varepsilon_0}{\alpha_3}\\[2mm] b=\dfrac{\alpha_1}{\alpha_3}\end{cases}\quad\text{or}\quad\begin{cases}\gamma=\dfrac{\beta_0}{\alpha_1}\\[2mm] \delta=\dfrac{\beta_2+\beta_1+\beta_0}{\alpha_3-\alpha_1}\\[2mm] \epsilon=\dfrac{\beta_2\alpha_1^2+\beta_1\alpha_1\alpha_3+\beta_0\alpha_3^2}{\alpha_3\alpha_1(\alpha_1-\alpha_3)}\\[2mm] \alpha=-n\\[2mm] \beta=\dfrac{\beta_2+(n-1)\alpha_3}{\alpha_3}\\[2mm] q=\dfrac{\varepsilon_0}{\alpha_3}\\[2mm] b=\dfrac{\alpha_1}{\alpha_3}\end{cases}$$

In either case, we have

$$\gamma+\delta+\epsilon=\alpha+\beta+1.$$

7.2.3 Subclass III: $\alpha_0=\alpha_1=\beta_0=0$

In this case, the second-order linear differential equation

$$x^2(\alpha_3 x+\alpha_2)\;y''(x)+x(\beta_2 x+\beta_1)\;y'-(\varepsilon_1 x+\varepsilon_0)\;y(x)=0, \tag{7.2.18}$$

has a polynomial solution of degree n if

$$\varepsilon_1=n(n-1)\;\alpha_3+n\;\beta_2,\quad n=0,1,2,\ldots, \tag{7.2.19}$$

along with the vanishing of $(n+1)\times(n+1)$-determinant Δ_{n+1} given by

$$\Delta_{n+1} = \begin{vmatrix} S_0 & & & & & & & \\ \gamma_1 & S_1 & & & & & & \\ & \gamma_2 & S_2 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \gamma_{n-2} & S_{n-2} & & & \\ & & & & \gamma_{n-1} & S_{n-1} & \\ & & & & & \gamma_n & S_n \end{vmatrix}$$

$$=\prod_{k=0}^{n} S_k=\prod_{k=0}^{n}(\varepsilon_0-k((k-1)\alpha_2+\beta_1)= \ 0$$

where all the other entries are zeros and

$$\begin{aligned} S_k &= \varepsilon_0 - k((k-1)\alpha_2+\beta_1) \\ \gamma_k &= \varepsilon_1-(k-1)((k-2)\alpha_3+\beta_2). \end{aligned} \tag{7.2.20}$$

Thus

$$\varepsilon_0 = k(k-1)\ \alpha_2 + k\ \beta_1, \quad k=0,1,2,\dots. \tag{7.2.21}$$

Consequently, the differential equation

$$\begin{aligned} &x^2(\alpha_3x+\alpha_2)\ y''(x)+x(\beta_2x+\beta_1)\ y'(x) \\ &\qquad -((n(n-1)\ \alpha_3+n\ \beta_2)x+(n(n-1)\ \alpha_2+n\ \beta_1))\ y(x)=0, \end{aligned} \tag{7.2.22}$$

has polynomial solutions as

$$y_n(x)=x^n, \quad n=0,1,2,\dots \tag{7.2.23}$$

Chapter 8

On the solutions of the differential equation: $(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4)\, y'' + (\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3)\, y' + (\varepsilon_0 + \varepsilon_1 x + \varepsilon_2 x^2)\, y = 0$

8.1 Series solution

Theorem 8.1.1. *The coefficients* $\{C_k\}_{k=0}^{\infty}$ *of the series solution* $y(x) = \sum_{k=0}^{\infty} C_k\, x^k$ *of the differential equation*

$$(\alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0) y''(x) + (\beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0) y'(x) \\ - (\varepsilon_2 x^2 + \varepsilon_1 x + \varepsilon_0) y(x) = 0, \tag{8.1.1}$$

satisfy the following five-term recurrence relation

$$(\alpha_4 (k-2)(k-3) + \beta_3 (k-2) - \varepsilon_2) C_{k-2} \\ + (\alpha_3 (k-1)(k-2) + \beta_2 (k-1) - \varepsilon_1) C_{k-1} + (\alpha_2 k(k-1) + \beta_1 k - \varepsilon_0) C_k \\ + (\alpha_1 k(k+1) + \beta_0 (k+1)) C_{k+1} + \alpha_0 (k+2)(k+1) C_{k+2} = 0. \tag{8.1.2}$$

where, by assumption, $C_{-2} = C_{-1} = 0$.

Proof: Substituting a power series

$$y(x) = \sum_{k=0}^{\infty} C_k\, x^k, \quad y'(x) = \sum_{k=1}^{\infty} k\, C_k r^{k-1}, \quad y''(x) = \sum_{k=2}^{\infty} k(k-1) C_k x^{k-2},$$

in the differential equation (8.1.1), and shifting the indices to make the summands appear in the form x^k, we obtain

$$-\varepsilon_0 C_0 + \beta_0 C_1 + 2\alpha_0 C_2 + (-\varepsilon_1 C_0 + (\beta_1 - \varepsilon_0) C_1 + 2(\alpha_1 + \beta_0) C_2 + 6\alpha_0 C_3) x \\ + (-\varepsilon_2 C_0 + (\beta_2 - \varepsilon_1) C_1 + (2\alpha_2 + 2\beta_1 - \varepsilon_0) C_2 + 3(2\alpha_1 + \beta_0) C_3 + 12\alpha_0 C_4) x^2 \\ + ((\beta_3 - \varepsilon_2) C_1 + (2\beta_2 - \varepsilon_1) C_2 + (6\alpha_2 + 3\beta_1 - \varepsilon_0) C_3 + 4(3\alpha_1 + \beta_0) C_4 + 20\alpha_0 C_5) x^3 \\ + \sum_{k=4}^{\infty} (\alpha_4 (k-2)(k-3) + \beta_3 (k-2) - \varepsilon_2) C_{k-2} + (\alpha_3 (k-1)(k-2) + \beta_2 (k-1) - \varepsilon_1) C_{k-1} \\ + (\alpha_2 k(k-1) + \beta_1 k - \varepsilon_0) C_k + (\alpha_1 k(k+1) + \beta_0 (k+1)) C_{k+1} + \alpha_0 (k+2)(k+1) C_{k+2}) x^k$$

From which we have, noting the coefficient of the variable x, that

$$
\begin{aligned}
-\varepsilon_0 C_0 + \beta_0 C_1 + 2\alpha_0 C_2 &= 0, \\
-\varepsilon_1 C_0 + (\beta_1 - \varepsilon_0) C_1 + 2(\alpha_1 + \beta_0) C_2 + 6\alpha_0 C_3 &= 0, \\
-\varepsilon_2 C_0 + (\beta_2 - \varepsilon_1) C_1 + (2\alpha_2 + 2\beta_1 - \varepsilon_0) C_2 + 3(2\alpha_1 + \beta_0) C_3 + 12\alpha_0 C_4 &= 0, \\
(\beta_3 - \varepsilon_2) C_1 + (2\beta_2 - \varepsilon_1) C_2 + (6\alpha_2 + 3\beta_1 - \varepsilon_0) C_3 + 4(3\alpha_1 + \beta_0) C_4 + 20\alpha_0 C_5 &= 0,
\end{aligned}
$$

and in general, we have the following five-term recurrence relation

$$
\begin{aligned}
&(\alpha_4(k-2)(k-3) + \beta_3(k-2) - \varepsilon_2) C_{k-2} \\
&\quad + (\alpha_3(k-1)(k-2) + \beta_2(k-1) - \varepsilon_1) C_{k-1} + (\alpha_2 k(k-1) + \beta_1 k - \varepsilon_0) C_k \\
&\quad + (\alpha_1 k(k+1) + \beta_0(k+1)) C_{k+1} + \alpha_0 (k+2)(k+1) C_{k+2} = 0. \qquad (8.1.3)
\end{aligned}
$$

8.2 Polynomial solutions

For polynomial solutions, we must have $c_{n+1} = c_{n+2} = c_{n+3} = c_{n+4} = 0$ and thus

$$\varepsilon_2 = n(n-1)\alpha_4 + n\beta_3,$$

where

$$
\begin{aligned}
&-\varepsilon_0 c_0 + \beta_0 c_1 + 2\alpha_0 c_2 = 0 \\
&-\varepsilon_1 c_0 + (\beta_1 - \varepsilon_0) c_1 + (2\alpha_1 + 2\beta_0) c_2 + 6\alpha_0 c_3 = 0 \\
&-\varepsilon_2 c_0 + (\beta_2 - \varepsilon_1) c_1 + (2\alpha_2 + 2\beta_1 - \varepsilon_0) c_2 + (6\alpha_1 + 3\beta_0) c_3 + 12\alpha_0 c_4 = 0 \\
&(\beta_3 - \varepsilon_2) c_1 + (2\alpha_3 + 2\beta_2 - \varepsilon_1) c_2 + (6\alpha_2 + 3\beta_1 - \varepsilon_0) c_3 + (12\alpha_1 + 4\beta_0) c_4 + 20\alpha_0 c_5 = 0 \\
&(2\alpha_4 + 2\beta_3 - \varepsilon_2) c_2 + (6\alpha_3 + 3\beta_2 - \varepsilon_1) c_3 + (12\alpha_2 + 4\beta_1 - \varepsilon_0) c_4 + (12\alpha_1 + 5\beta_0) c_5 + 30\alpha_0 c_6 = 0
\end{aligned}
$$

and so on. So for a zero-degree polynomial, we have

$$(\varepsilon_0)_{1\times 1} = 0, \quad c_0 \neq 0$$

for the first-degree polynomial solution, $c_n = 0, n \geq 2$, we have

$$
\begin{aligned}
&-\varepsilon_0 c_0 + \beta_0 c_1 = 0, \\
&-\varepsilon_1 c_0 + (\beta_1 - \varepsilon_0) c_1 = 0, \\
&-\varepsilon_2 c_0 + (\beta_2 - \varepsilon_1) c_1 = 0, \\
&(\beta_3 - \varepsilon_2) c_1 = 0.
\end{aligned}
$$

For $c_0 \neq 0$ and $c_1 \neq 0$, we must have

$$\varepsilon_2 = \beta_3$$

and

$$
\begin{aligned}
&-\varepsilon_0 c_0 + \beta_0 c_1 = 0 \\
&-\varepsilon_1 c_0 + (\beta_1 - \varepsilon_0) c_1 = 0 \\
&-\beta_3 c_0 + (\beta_2 - \varepsilon_1) c_1 = 0
\end{aligned}
$$

This is a system of linear equations of more unknowns than the number of the equations, for unique solution, we must have

$$
det \begin{pmatrix} -\varepsilon_0 & \beta_0 \\ -\varepsilon_1 & \beta_1 - \varepsilon_0 \end{pmatrix}_{2\times 2} = 0, \quad \text{and} \quad det \begin{pmatrix} -\varepsilon_0 & \beta_0 \\ -\beta_3 & \beta_2 - \varepsilon_1 \end{pmatrix}_{2\times 2} = 0, \quad c_0 \neq 0, \quad c_1 \neq 0
$$

for the second-degree polynomial solution, $c_n=0$ for $n\geq 3$, that is

$$\begin{aligned}
&-\varepsilon_0 c_0+\beta_0 c_1+2\alpha_0 c_2=0\\
&-\varepsilon_1 c_0+(\beta_1-\varepsilon_0)c_1+(2\alpha_1+2\beta_0)c_2=0\\
&-\varepsilon_2 c_0+(\beta_2-\varepsilon_1)c_1+(2\alpha_2+2\beta_1-\varepsilon_0)c_2=0\\
&(\beta_3-\varepsilon_2)c_1+(2\alpha_3+2\beta_2-\varepsilon_1)c_2=0\\
&(2\alpha_4+2\beta_3-\varepsilon_2)c_2=0
\end{aligned}$$

The last equation gives

$$\varepsilon_2=2\alpha_4+2\beta_3$$

and we now have

$$\begin{aligned}
&-\varepsilon_0 c_0+\beta_0 c_1+2\alpha_0 c_2=0\\
&-\varepsilon_1 c_0+(\beta_1-\varepsilon_0)c_1+(2\alpha_1+2\beta_0)c_2=0\\
&-(2\alpha_4+2\beta_3)c_0+(\beta_2-\varepsilon_1)c_1+(2\alpha_2+2\beta_1-\varepsilon_0)c_2=0\\
&(\beta_3-(2\alpha_4+2\beta_3))c_1+(2\alpha_3+2\beta_2-\varepsilon_1)c_2=0
\end{aligned}$$

Again, this is a system of linear equations with more unknowns than the number of equations. For unique solution, we must have for $c_0\neq 0,\ c_1\neq 0$, and $c_2\neq 0$ that

$$\det\begin{pmatrix}-\varepsilon_0 & \beta_0 & 2\alpha_0\\ -\varepsilon_1 & \beta_1-\varepsilon_0 & 2\alpha_1+2\beta_0\\ -(2\alpha_4+2\beta_3) & \beta_2-\varepsilon_1 & 2\alpha_2+2\beta_1-\varepsilon_0\end{pmatrix}_{3\times 3}=0,$$

and

$$\det\begin{pmatrix}-\varepsilon_0 & \beta_0 & 2\alpha_0\\ -\varepsilon_1 & \beta_1-\varepsilon_0 & 2\alpha_1+2\beta_0\\ 0 & \beta_3-(2\alpha_4+2\beta_3) & 6\alpha_3+3\beta_2-\varepsilon_1\end{pmatrix}_{3\times 3}=0.$$

For the third-degree polynomial solution, we have $c_n=0$ for $n\geq 4$, that is

$$\begin{aligned}
&-\varepsilon_0 c_0+\beta_0 c_1+2\alpha_0 c_2=0\\
&-\varepsilon_1 c_0+(\beta_1-\varepsilon_0)c_1+(2\alpha_1+2\beta_0)c_2+6\alpha_0 c_3=0\\
&-\varepsilon_2 c_0+(\beta_2-\varepsilon_1)c_1+(2\alpha_2+2\beta_1-\varepsilon_0)c_2+(6\alpha_1+3\beta_0)c_3=0\\
&(\beta_3-\varepsilon_2)c_1+(2\alpha_3+2\beta_2-\varepsilon_1)c_2+(6\alpha_2+3\beta_1-\varepsilon_0)c_3=0\\
&(2\alpha_4+2\beta_3-\varepsilon_2)c_2+(6\alpha_3+3\beta_2-\varepsilon_1)c_3=0\\
&(6\alpha_4+3\beta_1-\varepsilon_2)c_3=0.
\end{aligned}$$

The last equation gives

$$\varepsilon_2=6\alpha_4+3\beta_1,$$

and in this case

$$\begin{aligned}
&-\varepsilon_0 c_0+\beta_0 c_1+2\alpha_0 c_2=0\\
&-\varepsilon_1 c_0+(\beta_1-\varepsilon_0)c_1+(2\alpha_1+2\beta_0)c_2+6\alpha_0 c_3=0\\
&-(6\alpha_4+3\beta_1)c_0+(\beta_2-\varepsilon_1)c_1+(2\alpha_2+2\beta_1-\varepsilon_0)c_2+(6\alpha_1+3\beta_0)c_3=0\\
&(\beta_3-(6\alpha_4+3\beta_1))c_1+(2\alpha_3+2\beta_2-\varepsilon_1)c_2+(6\alpha_2+3\beta_1-\varepsilon_0)c_3=0
\end{aligned}$$

$$(2\alpha_4+2\beta_3-(6\alpha_4+3\beta_1))c_2+(6\alpha_3+3\beta_2-\varepsilon_1)c_3=0.$$

This is again a system of linear equations of more unknowns than the number of equations, thus we must have

$$\det\begin{pmatrix} -\varepsilon_0 & \beta_0 & 2\alpha_0 & 0 \\ -\varepsilon_1 & \beta_1-\varepsilon_0 & 2\alpha_1+2\beta_0 & 6\alpha_0 \\ -(6\alpha_4+3\beta_1) & \beta_2-\varepsilon_1 & 2\alpha_2+2\beta_1-\varepsilon_0 & 6\alpha_1+3\beta_0 \\ 0 & \beta_3-(6\alpha_4+3\beta_1) & 2\alpha_3+2\beta_2-\varepsilon_1 & 6\alpha_2+3\beta_1-\varepsilon_0 \end{pmatrix}_{4\times 4}=0$$

and

$$\det\begin{pmatrix} -\varepsilon_0 & \beta_0 & 2\alpha_0 & 0 \\ -(6\alpha_4+3\beta_1) & \beta_2-\varepsilon_1 & 2\alpha_2+2\beta_1-\varepsilon_0 & 6\alpha_1+3\beta_0 \\ 0 & \beta_3-(6\alpha_4+3\beta_1) & 2\alpha_3+2\beta_2-\varepsilon_1 & 6\alpha_2+3\beta_1-\varepsilon_0 \\ 0 & 0 & 2\alpha_4+2\beta_3-(6\alpha_4+3\beta_1) & 6\alpha_3+3\beta_2-\varepsilon_1 \end{pmatrix}_{4\times 4}=0.$$

In general, the necessary condition for polynomial solution of degree n is that

$$\varepsilon_2=n(n-1)\alpha_4+n\beta_3$$

and the sufficient condition is given by vanishing of the two $n+1\times n+1$ determinants

$$\begin{pmatrix} \beta_0 & \alpha_1 & \gamma_2 & & & & & & \\ \delta_0 & \beta_1 & \alpha_2 & \gamma_3 & & & & & \\ \rho_0 & \delta_1 & \beta_2 & \alpha_3 & \gamma_4 & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & \rho_{n-3} & \delta_{n-2} & \beta_{n-1} & \gamma_n \\ & & & & & & \rho_{n-2} & \delta_{n-1} & \beta_n \end{pmatrix}$$

and

$$\begin{pmatrix} \beta_0 & \alpha_1 & \gamma_2 & & & & & & \\ \delta_0 & \beta_1 & \alpha_2 & \gamma_3 & & & & & \\ \rho_0 & \delta_1 & \beta_2 & \alpha_3 & \gamma_4 & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & \rho_{n-3} & \delta_{n-2} & \beta_{n-1} & \gamma_n \\ & & & & & & \rho_{n-2} & \delta_{n-1} & \beta_n \end{pmatrix}$$

where

$$\begin{aligned} \beta_n &= n(n-1)\alpha_2+n\beta_1-\varepsilon_0 \\ \alpha_n &= n(n-1)\alpha_1+n\beta_0 \\ \gamma_n &= n(n-1)\alpha_0 \\ \delta_n &= n(n-1)\alpha_3+n\beta_2-\varepsilon_1 \\ \rho_n &= n(n-1)\alpha_4+n\beta_3-\varepsilon_2 \end{aligned}$$

Chapter 9

On the solutions of the differential equation: $x^2\left(\alpha_0+\alpha_1 x+\alpha_2 x^2\right) y''+x\left(\beta_0+\beta_1 x+\beta_2 x^2\right) y'+\left(\gamma_0+\gamma_1 x+\gamma_2 x^2\right) y=0$

9.1 Series solutions

Theorem 9.1.1. *The coefficients* $\{a_n\}_{n=0}^{\infty}$ *of the series solution* $y = x^r \sum_{n=0}^{\infty} a_n x^n$ *for the differential equation*

$$x^2(\alpha_0+\alpha_1 x+\alpha_2 x^2)y''+x(\beta_0+\beta_1 x+\beta_2 x^2)y'+(\gamma_0+\gamma_1 x+\gamma_2 x^2)y=0 \tag{9.1.1}$$

satisfies

$$\begin{gathered} a_0(r)=1, \qquad p_1(r)\,a_0(r)+p_0(r+1)\,a_1(r)=0, \\ p_0(n+r)a_n(r)+p_1(n+r-1)a_{n-1}(r)+p_2(n+r-2)a_{n-2}(r)=0, \quad n\geq 2, \end{gathered} \tag{9.1.2}$$

where

$$\begin{aligned} p_0(r)&=\alpha_0 r(r-1)+\beta_0 r+\gamma_0, \\ p_1(r)&=\alpha_1 r(r-1)+\beta_1 r+\gamma_1, \\ p_2(r)&=\alpha_2 r(r-1)+\beta_2 r+\gamma_2. \end{aligned}$$

Further, if r_1 *and* r_2 *are the real roots of* $\alpha_0+\alpha_1 x+\alpha_2 x^2=0$*, then* $x\in(0,\min(r_1,r_2))$*.*

Proof. Direct differentiation of the series solution $y=\sum_{n=0}^{\infty} a_n x^{n+r}$ yields

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}, \qquad y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}.$$

Multiplying y' by x and y'' by x^2 yields

$$xy'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}, \qquad x^2y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}.$$

Therefore

$$\alpha x^2y''+\beta xy'+\gamma y=\sum_{n=0}^{\infty}[(n+r-1)\alpha(n+r)+\beta(n+r)+\gamma]a_n x^{n+r}$$

$$=\sum_{n=0}^{\infty} p(n+r) a_n x^{n+r}. \tag{9.1.3}$$

where $p(n)=(n-1)\alpha(n)+\beta(n)+\gamma$. Multiplying (9.1.3) by x yields

$$x(\alpha x^2 y''+\beta x y'+\gamma y)=\sum_{n=0}^{\infty} p(n+r) a_n x^{n+r+1}=\sum_{n=1}^{\infty} p(n+r-1) a_{n-1} x^{n+r}. \tag{9.1.4}$$

In addition multiplying (9.1.3) by x^2 yields

$$x^2(\alpha x^2 y''+\beta x y'+\gamma y)=\sum_{n=0}^{\infty} p(n+r) a_n x^{n+r+2}=\sum_{n=2}^{\infty} p(n+r-2) a_{n-2} x^{n+r}. \tag{9.1.5}$$

Writing equation (9.1.1) as

$$(\alpha_0 x^2 y''+\beta_0 x y'+\gamma_0 y)+x(\alpha_1 x^2 y''+\beta_1 x y'+\gamma_1 y)+x^2(\alpha_2 x^2 y''+\beta_2 x y'+\gamma_2 y)=0, \tag{9.1.6}$$

it follows, using

$$\begin{aligned}
\alpha_0 x^2 y''+\beta_0 x y'+\gamma_0 y &= \sum_{n=0}^{\infty} p_0(n+r) a_n x^{n+r}\\
x(\alpha_1 x^2 y''+\beta_1 x y'+\gamma_1 y) &= \sum_{n=1}^{\infty} p_1(n+r-1) a_{n-1} x^{n+r},\\
x^2(\alpha_2 x^2 y''+\beta_2 x y'+\gamma_2 y) &= \sum_{n=2}^{\infty} p_2(n+r-2) a_{n-2} x^{n+r},
\end{aligned}$$

that

$$\sum_{n=0}^{\infty} p_0(n+r) a_n x^{n+r}+\sum_{n=1}^{\infty} p_1(n+r-1) a_{n-1} x^{n+r}+\sum_{n=2}^{\infty} p_2(n+r-2) a_{n-2} x^{n+r}=0$$

whence

$$\begin{aligned}
& p_0(r) a_0 x^r+[p_0(r+1) a_1+p_1(r) a_0] x^{r+1}\\
+&\sum_{n=2}^{\infty}[p_0(n+r) a_n+p_1(n+r-1) a_{n-1}+p_2(n+r-2) a_{n-2}] x^{n+r}=0.
\end{aligned}$$

□

Theorem 9.1.2. *Let r_1 and r_2 be the real roots of the indicial equation $p_0(r)=0$ where r_1 and r_2, where $r_1 \geq r_2$. Then*

$$y_1(x)=y(x, r_1)=x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n \tag{9.1.7}$$

is a Frobenius solution of differential equation (9.1.1) *namely*

$$x^2(\alpha_0+\alpha_1 x+\alpha_2 x^2) y''+x(\beta_0+\beta_1 x+\beta_2 x^2) y'+(\gamma_0+\gamma_1 x+\gamma_2 x^2) y=0.$$

Moreover, if r_1-r_2 is not an integer then

$$y_2(x)=y(x, r_2)=x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n \tag{9.1.8}$$

is also the second linearly independent Frobenius solution, in other words $\{y_1, y_2\}$ is a fundamental set of solutions.

9.1.1 Subclass I: $x^2\left(\alpha_0+\alpha_1x\right)y''+x\left(\beta_0+\beta_1x\right)y'+\left(\gamma_0+\gamma_1x\right)y=0.$

Theorem 9.1.3. *The coefficients* $\{a_n\}_{n=0}^{\infty}$ *of the series solution* $y=x^r\sum_{n=0}^{\infty}a_nx^n$ *for the differential equation*

$$x^2\left(\alpha_0+\alpha_1x\right)y''+x\left(\beta_0+\beta_1x\right)y'+\left(\gamma_0+\gamma_1x\right)y=0, \tag{9.1.9}$$

satisfy the two-term recurrence relation

$$a_0(r)=1,\qquad a_n(r)=-\frac{p_1(n+r-1)a_{n-1}(r)}{p_0(n+r)},\quad n\geq 2, \tag{9.1.10}$$

where

$$p_0(r)=\alpha_0r(r-1)+\beta_0r+\gamma_0,\qquad p_1(r)=\alpha_1r(r-1)+\beta_1r+\gamma_1.$$

The exact solution is given in terms of the generalized hypergeometric function

$$y(x)=x^r\times$$
$$ {}_3F_2\left(1,\frac{\alpha_1(3-2n-2r)-\beta_1+\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1},\frac{\alpha_1(3-2n-2r)-\beta_1-\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1};\right.$$
$$\left.\frac{\alpha_0(5-2n-2r)-\beta_0+\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0},\frac{\alpha_1(5-2n-2r)-\beta_0-\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0};-\frac{\alpha_1}{\alpha_0}x\right). \tag{9.1.11}$$

Proof. For the given equation

$$x^2(\alpha_0+\alpha_1x)y''+x(\beta_0+\beta_1x)y'+(\gamma_0+\gamma_1x)y=0.$$

with $\alpha_0\neq 0$ we set, $\alpha_2=\beta_2=\gamma_2=0$, in equation (9.1.1), therefore $p_2\equiv 0$ and the recurrence relations (9.1.2) reduce to

$$a_0(r)=1,\qquad a_n(r)=-\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r),\qquad n\geq 1.$$

The recurrence relation implies

$$\begin{aligned}a_n(r)&=(-1)^2\,\frac{p_1(n+r-1)}{p_0(n+r)}\cdot\frac{p_1(n+r-2)}{p_0(n+r-1)}a_{n-2}(r)\\&=(-1)^n\,\frac{p_1(n+r-1)}{p_0(n+r)}\cdot\frac{p_1(n+r-2)}{p_0(n+r-1)}\cdot\frac{p_1(n+r-3)}{p_0(n+r-2)}\cdots\frac{p_1(r)}{p_0(r+1)}.\end{aligned}$$

By means of the Pochhammer symbol

$$(\alpha)_n=\alpha(\alpha+1)\ldots(\alpha+n-1)=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{9.1.12}$$

we may write

$$a_n(r)=\left(-\frac{\alpha_1}{\alpha_0}\right)^n$$

$$\frac{\left(\frac{\alpha_1(3-2n-2r)-\beta_1+\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1}\right)_n\left(\frac{\alpha_1(3-2n-2r)-\beta_1-\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1}\right)_n}{\left(\frac{\alpha_0(5-2n-2r)-\beta_0+\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0}\right)_n\left(\frac{\alpha_1(5-2n-2r)-\beta_0-\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0}\right)_n}.$$

The solution can be written then as

$$y=x^r\sum_{n=0}^{\infty}\left(-\frac{\alpha_1}{\alpha_0}\,x\right)^n$$

$$\frac{\left(\frac{\alpha_1(3-2n-2r)-\beta_1+\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1}\right)_n\left(\frac{\alpha_1(3-2n-2r)-\beta_1-\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1}\right)_n}{\left(\frac{\alpha_0(5-2n-2r)-\beta_0+\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0}\right)_n\left(\frac{\alpha_1(5-2n-2r)-\beta_0-\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0}\right)_n}$$

$$=x^r\sum_{n=0}^{\infty}\left(-\frac{\alpha_1}{\alpha_0}\,x\right)^n\frac{(1)_n}{n!}$$

$$\frac{\left(\frac{\alpha_1(3-2n-2r)-\beta_1+\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1}\right)_n\left(\frac{\alpha_1(3-2n-2r)-\beta_1-\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1}\right)_n}{\left(\frac{\alpha_0(5-2n-2r)-\beta_0+\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0}\right)_n\left(\frac{\alpha_1(5-2n-2r)-\beta_0-\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0}\right)_n}$$

$$=x^r\times$$

$$ {}_3F_1\left(1,\frac{\alpha_1(3-2n-2r)-\beta_1+\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1},\frac{\alpha_1(3-2n-2r)-\beta_1-\sqrt{\alpha_1-\beta_1)^2-4\alpha_1\gamma_1}}{2\alpha_1};\right.$$

$$\left.\frac{\alpha_0(5-2n-2r)-\beta_0+\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0},\frac{\alpha_1(5-2n-2r)-\beta_0-\sqrt{\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{2\alpha_0};-\frac{\alpha_1}{\alpha_0}\,x\right).$$

□

9.1.2 Subclass II: $x^2(\alpha_0+\alpha_2x^2)y''+x(\beta_0+\beta_2x^2)y'+(\gamma_0+\gamma_2x^2)y=0$.

Theorem 9.1.4. *The coefficients* $\{a_n\}_{n=0}^{\infty}$ *of the series solution* $y=x^r\sum_{n=0}^{\infty}a_nx^n$ *for the differential equation*

$$x^2(\alpha_0+\alpha_2x^2)y''+x(\beta_0+\beta_2x^2)y'+(\gamma_0+\gamma_2x^2)y=0 \tag{9.1.13}$$

satisfy the two-term recurrence relation

$$a_0(r)=1,a_1(r)=0,\quad a_n(r)=-\frac{p_2(n+r-2)}{p_0(n+r)}a_{n-2}(r),\quad n\ge 2, \tag{9.1.14}$$

where

$$p_0(r)=\alpha_0r(r-1)+\beta_0r+\gamma_0,\qquad p_2(r)=\alpha_2r(r-1)+\beta_2r+\gamma_2.$$

The exact solutions are given in terms of the generalized hypergeometric function

$$\begin{aligned}y(x) =& x^r\,{}_3F_2\Bigg(1, \frac{r}{2}+\frac{\beta_2}{4\alpha_2}-\frac{1}{4}-\frac{\sqrt{(\alpha_2-\beta_2)^2-4\alpha_2\gamma_2}}{4\alpha_2}, \frac{r}{2}+\frac{\beta_2}{4\alpha_2}+\frac{\sqrt{(\alpha_2-\beta_2)^2-4\alpha_2\gamma_2}}{4\alpha_2}-\frac{1}{4};\\ &\frac{r}{2}+\frac{\beta_0}{4\alpha_0}+\frac{3}{4}-\frac{\sqrt{(\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{4\alpha_0}, \frac{r}{2}+\frac{\beta_0}{4\alpha_0}+\frac{\sqrt{(\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{4\alpha_0}+\frac{3}{4}; -\frac{x^2\alpha_2}{\alpha_0}\Bigg)\end{aligned} \tag{9.1.15}$$

Proof. Consider

$$x^2(\alpha_0+\alpha_2x^2)y''+x(\beta_0+\beta_2x^2)y'+(\gamma_0+\gamma_2x^2)y=0,$$

with $\alpha_0 \neq 0$. For this equation, $\alpha_1=\beta_1=\gamma_1=0$, so $p_1\equiv 0$ and the recurrence relations simplify to

$$a_0(r)=1,\quad a_1(r)=0,\quad a_n(r)=-\frac{p_2(n+r-2)}{p_0(n+r)}a_{n-2}(r),\qquad n\geq 2.$$

Since $a_1(r)=0$, the last equation implies that $a_n(r)=0$ if n is odd, so the Frobenius solutions are of the form

$$y(x,r)=x^r\sum_{m=0}^{\infty}a_{2m}(r)x^{2m}$$

where

$$a_0(r)=1,\quad a_{2m}(r)=-\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r),\qquad m\geq 1.$$

Since

$$a_{2m}(r)=-\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r),$$

recursively

$$\begin{aligned}a_{2m}(r)&=(-1)^2\frac{p_2(2m+r-2)p_2(2m+r-4)}{p_0(2m+r)p_0(2m+r-2)}a_{2m-4}(r)\\ &=(-1)^3\frac{p_2(2m+r-2)p_2(2m+r-4)p_2(2m+r-6)}{p_0(2m+r)p_0(2m+r-2)p_0(2m+r-4)}a_{2m-6}(r)\\ &=(-1)^4\frac{p_2(2m+r-2)p_2(2m+r-4)p_2(2m+r-6)p_2(2m+r-8)}{p_0(2m+r)p_0(2m+r-2)p_0(2m+r-4)p_0(2m+r-6)}a_{2m-8}(r)\\ &=(-1)^m\prod_{j=1}^{m}\frac{p_2(2m+r-2j)}{p_0(2m+r+2(1-j))}a_0=(-1)^m\prod_{j=1}^{m}\frac{p_2(r+2j-2)}{p_0(r+2j)}a_0,\end{aligned}$$

where we used

$$\begin{aligned}j=m-k+1,\quad 1\leq j\leq m,\quad 1\leq m-k+1\leq m,\quad 1-m-1\leq -k\leq -m-1+m,\\ -m\leq -k\leq -1,\\ m\geq k\geq 1.\end{aligned}$$

Thus,

$$y(x,r)=x^r\sum_{m=0}^{\infty}(-1)^m\prod_{j=1}^{m}\frac{p_2(r+2j-2)}{p_0(r+2j)}x^{2m},$$

which yields

$$
\begin{aligned}
y(x,r) &= x^r \sum_{m=0}^{\infty}(-1)^m \prod_{j=1}^{m} \frac{\alpha_2(r+2j-2)(r+2j-2-1)+\beta(r+2j-2)+\gamma_2}{a_0(r+2j)(r+2j-1)+\beta_0(r+2j)+\gamma_0} x^{2m} \\
&= x^r\, {}_3F_2\Bigg(1, \frac{r}{2}+\frac{\beta_2}{4\alpha_2}-\frac{1}{4}-\frac{\sqrt{(\alpha_2-\beta_2)^2-4\alpha_2\gamma_2}}{4\alpha_2}, \frac{r}{2}+\frac{\beta_2}{4\alpha_2}+\frac{\sqrt{(\alpha_2-\beta_2)^2-4\alpha_2\gamma_2}}{4\alpha_2}-\frac{1}{4}\,; \\
&\quad \frac{r}{2}+\frac{\beta_0}{4\alpha_0}+\frac{3}{4}-\frac{\sqrt{(\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{4\alpha_0}, \frac{r}{2}+\frac{\beta_0}{4\alpha_0}+\frac{\sqrt{(\alpha_0-\beta_0)^2-4\alpha_0\gamma_0}}{4\alpha_0}+\frac{3}{4}; -\frac{x^2\alpha_2}{\alpha_0}\Bigg).
\end{aligned}
$$

$\square$

Chapter 10

Conclusion and future work

In this work, we have categorized several classes of second-order differential equations and their exact solutions. Five classes of differential equations have been investigated:

- First class (Chapter 5):

$$(1 + \alpha\, x + \beta\, x^2)\, y'' + (\gamma + \delta\, x)\, y' + \varepsilon\, y = 0.$$

- Second class (Chapter 6):

$$x\, (\alpha_1 + \alpha_2\, x)\, y'' + (\beta_0 + \beta_1\, x)\, y' + \varepsilon\, y = 0.$$

- Third class (Chapter 7):

$$(\alpha_0 + \alpha_1\, x + \alpha_2\, x^2 + \alpha_3 x^3)\, y'' + (\beta_0 + \beta_1\, x + \beta_2\, x^2)\, y' + (\varepsilon_0 + \varepsilon_1\, x)\, y = 0.$$

- Fourth class (Chapter 8):

$$\begin{aligned}(\alpha_0 + \alpha_1\, x + \alpha_2\, x^2 + \alpha_3 x^3 + \alpha_4 x^4)\, y'' &+ (\beta_0 + \beta_1\, x + \beta_2\, x^2 + \beta_3\, x^3)\, y' \\ &+ (\varepsilon_0 + \varepsilon_1\, x + \varepsilon_2\, x^2)\, y = 0.\end{aligned}$$

- Fifth class (Chapter 9):

$$x^2\, (\alpha_0 + \alpha_1 x + \alpha_2 x^2)\, y'' + x\, (\beta_0 + \beta_1 x + \beta_2 x^2)\, y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)\, y = 0.$$

For each of these classes, the general series solutions were developed and the conditions under which polynomial solutions are existed were also reported. Explicit evaluation of their particular solutions were obtained in forms suitable for direct use in physics and engineering applications.

The extension of the present work to higher-order differential equations is a subject for future investigation. The classification of all the polynomial solutions and their possible connection with the classical orthogonal polynomials is another direction worth studying.

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